

The Uniqueness of the Spectral Flow on Spaces of Unbounded Self-adjoint Fredholm Operators

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ABSTRACT. We discuss several natural metrics on spaces of unbounded self-adjoint operators and their relations, among them the Riesz and the graph metric. We show that the topologies of the spaces of Fredholm operators resp. invertible operators depend heavily on the metric. Nevertheless we prove that in all cases the spectral flow is up to a normalization the only integer invariant of non-closed paths which is path additive and stable under homotopies with endpoints varying in the space of invertible self-adjoint operators.

Furthermore we show that for certain Riesz continuous paths of self-adjoint Fredholm operators the spectral flow can be expressed in terms of the index of the pair of positive spectral projections at the endpoints.

Finally we review the Cordes–Labrousse theorem on the stability of the Fredholm index with respect to the graph metric in a modern language and we generalize it to the Clifford index and to the equivariant index.

1. Introduction

Let H be a separable complex Hilbert space. Then it is well-known that the space $\mathcal{BF}^{\text{sa}} = \mathcal{BF}^{\text{sa}}(H)$ of bounded self-adjoint Fredholm operators has three connected components, i.e. \mathcal{BF}^{sa} is the disjoint union

$$(1.1) \quad \mathcal{BF}^{\text{sa}} = \mathcal{BF}_+^{\text{sa}} \cup \mathcal{BF}_-^{\text{sa}} \cup \mathcal{BF}_*^{\text{sa}},$$

where $\mathcal{BF}_\pm^{\text{sa}}$ denote the subspaces of essentially positive/negative operators and $\mathcal{BF}_*^{\text{sa}} = \mathcal{BF}^{\text{sa}} \setminus (\mathcal{BF}_+^{\text{sa}} \cup \mathcal{BF}_-^{\text{sa}})$.

$\mathcal{BF}_\pm^{\text{sa}}$ are trivially contractible. Atiyah and Singer [AS69] showed that the interesting component $\mathcal{BF}_*^{\text{sa}}$ is a classifying space for the K^1 -functor. In particular, one has

$$(1.2) \quad \pi_k(\mathcal{BF}_*^{\text{sa}}, I) \simeq \begin{cases} \mathbb{Z}, & k \text{ odd,} \\ 0, & k \text{ even.} \end{cases}$$

The isomorphism

$$(1.3) \quad \text{SF} : \pi_1(\mathcal{BF}_*^{\text{sa}}, I) \longrightarrow \mathbb{Z}$$

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is the celebrated *spectral flow*, although it was not addressed as such in loc. cit. The spectral flow was introduced and generalized to non-closed paths in the famous series of papers on spectral asymmetry by Atiyah, Patodi, and Singer [APS75]. In the finite-dimensional context the spectral flow probably dates even back to Morse and his index theorem.

It is impossible to give a complete account on the literature about the spectral flow. I would like to emphasize, however, that a rigorous definition of the spectral flow for (non-closed) continuous paths of bounded self-adjoint Fredholm operators is non-trivial. After the intuitively appealing approach of [APS75] the spectral flow was folklore and people did not feel the need or found it too trivial to bother about the definition and its basic properties. J. Phillips [Phi96] presented a completely different rigorous approach to the spectral flow of bounded self-adjoint Fredholm operators. Let us briefly summarize his definition:

DEFINITION 1.1. Let $f : [0, 1] \rightarrow \mathcal{BF}^{\text{sa}}$ be a continuous path of bounded self-adjoint Fredholm operators. Choose a subdivision $0 = t_0 < t_1 < \dots < t_n = 1$ of the interval such that there exist $\varepsilon_j > 0, j = 1, \dots, n$ with $\pm\varepsilon_j \notin \text{spec } f(t)$ and $[-\varepsilon_j, \varepsilon_j] \cap \text{spec}_{\text{ess}} f(t) = \emptyset$ for $t_{j-1} \leq t \leq t_j$. Then the *spectral flow* of f is defined by

$$\text{SF}(f) := \sum_{j=1}^n \left(\text{rank}(1_{[0, \varepsilon_j)}(f(t_j))) - \text{rank}(1_{[0, \varepsilon_j)}(f(t_{j-1}))) \right).$$

Here we have used the following notation which will be in effect throughout the paper: by 1_X we denote the characteristic function of X and for a Borel subset $X \subset \mathbb{C}$ and a normal operator T we denote by $1_X(T)$ the normal operator obtained by plugging T into 1_X via the Borel functional calculus.

It is shown in [Phi96] that a subdivision with the desired properties indeed exists and that SF is well-defined, path additive, and homotopy invariant. Also the isomorphism (1.3) is reproved.

It is an easy consequence of the isomorphism (1.3) that the spectral flow is up to normalization the only path additive and homotopy invariant integer-valued function from paths of self-adjoint Fredholm operators (see Theorem 5.4 below for a precise formulation).

In various branches of mathematics the spectral flow of families of *unbounded* operators arises naturally (e.g. in Floer homology, Nicolaescu [Nic95], Robbin and Salamon [RS95] to mention only a few). In the case of boundary value problems one even has to deal with operators with varying domains. Superficially, one might be tempted to believe that the aforementioned results for bounded operators just carry over with only minor modifications.

(Un)fortunately, this is not the case. So, denote by \mathcal{C}^{sa} the set of possibly unbounded self-adjoint operators in H and by $\mathcal{CF}^{\text{sa}} \subset \mathcal{C}^{\text{sa}}$ the subspace of (unbounded) self-adjoint Fredholm operators. At first, there exist several natural metrics on (subspaces of) \mathcal{C}^{sa} and results may depend on the metric. The weakest metric is the graph or gap metric, d_G , which was studied systematically by Cordes and Labrousse [CL63]. Another metric is the Riesz metric, d_R , which was discussed by Nicolaescu in the unpublished note [Nic00].

If one considers only operators with a fixed domain there is even another metric: let D be a fixed self-adjoint operator with domain $W := \mathcal{D}(D)$. On the space $\mathcal{B}^{\text{sa}}(W, H) := \{T \in \mathcal{C}^{\text{sa}} \mid \mathcal{D}(T) = W\}$ there is another natural metric d_W (see

Def. 2.1). For d_W -continuous paths $f : [0, 1] \rightarrow \mathcal{B}^{\text{sa}}(W, H)$ of Fredholm operators the spectral flow was defined by Booß-Bavnbek and Furutani [BBF98] (with the additional assumption $f(t) - f(0)$ bounded) and in [RS95] (with the assumption that D has compact resolvent). Moreover in [RS95, Sec. 4] it was shown that in their case the spectral flow is also unique in the sense described above.

For the Riesz metric the results mentioned at the beginning of this section indeed carry over verbatim. Namely, in subsection 5.4 we will show that the natural inclusion of the pair $(\mathcal{B}\mathcal{F}^{\text{sa}}, G\mathcal{B}^{\text{sa}})$ into $(\mathcal{C}\mathcal{F}^{\text{sa}}, G\mathcal{C}^{\text{sa}}, d_R)$ is indeed a homotopy equivalence (GX denotes the invertible elements in X). Hence the unbounded analogue of $\mathcal{B}\mathcal{F}_*^{\text{sa}}$, $(\mathcal{C}\mathcal{F}_*^{\text{sa}}, d_R)$ is a classifying space for the K^1 -functor and the analogue of (1.3) holds. There is a drawback: the Riesz topology is so strong that it is hard to prove continuity of maps into $(\mathcal{C}^{\text{sa}}, d_R)$. As an example consider a compact manifold with boundary, M , and a Dirac operator D on M . Elliptic boundary conditions for D are parametrized by certain pseudodifferential projections (cf. e.g. Brüning and Lesch [BL01] and references therein) P on the boundary. Denote by D_P the self-adjoint realization of D with boundary condition P . Then $P \mapsto D_P$ is graph continuous. Note that $P \mapsto D_P$ is a family of operators with varying domains! See Booß-Bavnbek, Lesch, and Phillips [BBLP01, Sec. 3] for details. It is not known, at least not to the author, whether $P \mapsto D_P$ is Riesz continuous or not.

In the development of the spectral flow for paths of unbounded operators one first tried to use the Riesz metric. But continuity proofs for simple maps like $\mathcal{B}^{\text{sa}} \rightarrow \mathcal{C}^{\text{sa}}, A \mapsto D + A$ (D a fixed self-adjoint operator) are rather complicated (cf. [Phi97, Thm. A.8], [BBF98, Sec. 4] and Proposition 2.2).

A drawback (if it is one!) of the weaker graph topology is that the homotopy type $\mathcal{C}\mathcal{F}^{\text{sa}}$ is presumably more complicated. For graph continuous paths in $\mathcal{C}\mathcal{F}^{\text{sa}}$ it was shown in [BBLP01] that the definition of the spectral flow in [Phi96] carries over. More importantly, an alternative definition of the spectral flow in terms of the Cayley transform and the classical winding number is given. This uses results of Kirk and Lesch [KL00, Sec. 6] which show that the spectral flow, the Maslov index, and the classical winding number are intimately connected.

It should come as a surprise that, as opposed to (1.1), $(\mathcal{C}\mathcal{F}^{\text{sa}}, d_G)$ is path connected [BBLP01, Thm. 1.10]. In light of this one should also be ready for surprises concerning the fundamental group. Still, except that the spectral flow is a surjective homomorphism from $\pi_1(\mathcal{C}\mathcal{F}^{\text{sa}}, d_G)$ onto the integers nothing about $\pi_1(\mathcal{C}\mathcal{F}^{\text{sa}}, d_G)$ is known. Therefore, we single out the following open problem:

PROBLEM. Find $\pi_1(\mathcal{C}\mathcal{F}^{\text{sa}}, d_G)$. Even more, is $(\mathcal{C}\mathcal{F}^{\text{sa}}, d_G)$ a classifying space for the K^1 -functor?

We do not have a good guess for the answer to this problem and therefore we do not further speculate.

Since the fundamental group of $(\mathcal{C}\mathcal{F}^{\text{sa}}, d_G)$ is not known the uniqueness of the spectral flow on $(\mathcal{C}\mathcal{F}^{\text{sa}}, d_G)$ cannot (yet) be proved along the lines of the case of bounded operators. The current paper wants, among other things, to fill this gap and prove that as in the bounded case the spectral flow is up to normalization the only path additive and homotopy invariant integer-valued function from paths in $(\mathcal{C}\mathcal{F}^{\text{sa}}, d_G)$ (see Theorem 5.9 below for a precise formulation).

The second goal of this paper is to give an account on the various metrics on (subspaces of) \mathcal{C}^{sa} and their relations. It should be noted at this point that the

restriction to self-adjoint operators is not a loss of generality as far as the space of *all* unbounded operators is concerned. Namely, denote by \mathcal{C} the set of closed densely defined operators in H . Then the map

$$(1.4) \quad T \mapsto \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

is a natural embedding $\mathcal{C}(H) \hookrightarrow \mathcal{C}^{\text{sa}}(H \oplus H)$. So each metric on $\mathcal{C}^{\text{sa}}(H \oplus H)$ naturally induces a metric on $\mathcal{C}(H)$ and, obviously, each metric on $\mathcal{C}(H)$ induces one on $\mathcal{C}^{\text{sa}}(H)$. That is the reason why we restrict ourselves to the discussion of self-adjoint operators. This approach, admittedly, does not cover all cases one could think of in this context: for example the space of all closed operators with a fixed domain (for instance the Sobolev space $H^1 \subset L^2$ on a manifold) does not seem to be treatable by this approach; see however Definition 2.1 below.

The paper is organized as follows:

In Section 2 we introduce the various metrics on spaces of unbounded operators, namely the graph metric d_G , the Riesz metric d_R , the d_W -metric and the norm metric d_N . The latter two are defined only on certain subspaces of \mathcal{C}^{sa} . We show that $d_N \gneq d_W \gneq d_R \gneq d_G$ (see Propositions 2.2 and 2.4), i.e. d_N is strictly stronger than d_W and so on. We put some effort in the construction of counterexamples which show that the metrics induce different topologies even on relatively "small" subsets of \mathcal{C}^{sa} .

Section 3 discusses the relation between the spectral flow and the index of a pair of projections. More precisely, we show that for a *Riesz* continuous path T_t of self-adjoint Fredholm operators with the additional property that the domains are fixed and that $T_t - T_0$ is compact the spectral flow is the index of the pair of positive spectral projections at the endpoints (Theorem 3.6). This generalizes work of Bunke [Bun94] who has considered the special case of families of the form $D_t := D + tR$ where D is a Dirac operator on a compact manifold and R is a self-adjoint bundle endomorphism satisfying additional assumptions.

As an application we prove an abstract Toeplitz index theorem (Proposition 3.9).

The positive spectral projections and their relative index were used to define the spectral flow even in the von Neumann algebra context by Phillips [Phi97].

Section 4 presents the results of the celebrated paper by Cordes and Labrousse [CL63] in modern language and in a very concise form. We go slightly beyond loc. cit. and prove the stability of the Clifford index and the G -index with respect to the graph metric (Theorem 4.3). Moreover we prove that with respect to both, d_G and d_R , the bounded self-adjoint operators are open and dense in \mathcal{C}^{sa} and that d_G and d_R induce the norm topology on bounded operators (Proposition 4.1).

Section 5 is the heart of the paper. We show that the spectral flow can be characterized axiomatically, i.e. it is the only integer invariant of continuous paths of self-adjoint Fredholm operators which satisfies *Homotopy*, *Concatenation*, and *Normalization*. We prove this uniqueness in the finite-dimensional case, the bounded case, for the graph and Riesz metric, and for the d_W -metric (Theorems 5.4, 5.7, 5.9, 5.10, and 5.13). On the way we generalize the method of [BBLP01] to show that the space of invertible elements of $(\mathcal{C}^{\text{sa}}, d_G)$ is still path connected (Proposition 5.8).

In the Appendix we finally collect a few useful operator estimates which we need and which did not quite fit into the course of the paper.

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2. Topologies on spaces of unbounded self-adjoint operators

In this section we will discuss various natural metrics and their topologies on spaces of unbounded operators. These are used frequently in the literature. However, a systematic comparison does not seem to be available except for the Riesz and the graph topology, see e.g. [Nic00], [BBLP01].

Notations. Let H be a separable complex Hilbert space. First let us introduce some notation for various spaces of operators in H :

- $\mathcal{C}(H)$ closed densely defined operators in H ,
- $\mathcal{B}(H)$ bounded linear operators $H \rightarrow H$,
- $\mathcal{U}(H)$ unitary operators $H \rightarrow H$,
- $\mathcal{K}(H)$ compact linear operators $H \rightarrow H$,
- $\mathcal{BF}(H)$ bounded Fredholm operators $H \rightarrow H$,
- $\mathcal{CF}(H)$ (closed) densely defined Fredholm operators in H .

If no confusion is possible we will omit " (H) " and write $\mathcal{C}, \mathcal{B}, \mathcal{K}$ etc. By $\mathcal{C}^{\text{sa}}, \mathcal{B}^{\text{sa}}$ etc. we denote the set of self-adjoint elements in \mathcal{C}, \mathcal{B} etc.

\mathcal{C}^{sa} carries two natural metrics, the *Riesz metric* and the *gap or graph metric*. The Riesz metric is given by

$$(2.1) \quad d_R(T_1, T_2) := \|F(T_1) - F(T_2)\|,$$

where

$$(2.2) \quad F(T) := T(I + T^2)^{-1/2}.$$

The graph metric is given by

$$(2.3) \quad d_G(T_1, T_2) := \frac{1}{2} \|\kappa(T_1) - \kappa(T_2)\| = \|(T_1 + i)^{-1} - (T_2 + i)^{-1}\|,$$

where

$$(2.4) \quad \kappa(T) = (T - i)(T + i)^{-1}$$

is the Cayley transform (cf. [BBLP01, Thm. 1.1]). For alternative descriptions of the graph metric see Nicolaescu [Nic97, Appendix A] or Kato [Kat76].

If we restrict ourselves to operators with a fixed domain then there are even more metrics: let D be a fixed self-adjoint operator in H . The domain of D , $W := \mathcal{D}(D)$, equipped with the graph scalar product,

$$(2.5) \quad \langle x, y \rangle_D := \langle x, y \rangle + \langle Dx, Dy \rangle,$$

is then a Hilbert space which is continuously embedded in H .

DEFINITION 2.1. $\mathcal{B}^{\text{sa}}(W, H) := \{T \in \mathcal{C}^{\text{sa}} \mid \mathcal{D}(T) = W\}$. $\mathcal{B}^{\text{sa}}(W, H)$ is equipped with the metric

$$d_W(T_1, T_2) := \|T_1 - T_2\|_{W \rightarrow H} = \|(T_1 - T_2)(I + D^2)^{-1/2}\|_{H \rightarrow H}.$$

Note that if $T_1 \in \mathcal{B}^{\text{sa}}(W, H)$ is invertible as an element of \mathcal{C}^{sa} then T_1^{-1} maps H continuously into W and thus for any $T_2 \in \mathcal{B}^{\text{sa}}(W, H)$ the operator $T_2 T_1^{-1}$ is a bounded operator $H \rightarrow H$. This will be used in the sequel without further notice.

On the subspace

$$(2.6) \quad D + \mathcal{B}^{\text{sa}} = \{D + C \mid C \in \mathcal{B}^{\text{sa}}\} \simeq \mathcal{B}^{\text{sa}}$$

we have additionally the norm distance

$$(2.7) \quad d_N(D + C_1, D + C_2) = \|C_1 - C_2\|.$$

PROPOSITION 2.2. *The natural maps*

$$\begin{aligned} (\mathcal{B}^{\text{sa}}, d_N) &\xrightarrow{\alpha} (\mathcal{B}^{\text{sa}}(W, H), d_W) \xrightarrow{\beta} (\mathcal{C}^{\text{sa}}, d_R) \xrightarrow{\text{id}} (\mathcal{C}^{\text{sa}}, d_G) \\ C &\longmapsto D + C \end{aligned}$$

are continuous. Here β is the natural inclusion.

REMARK 2.3.

(1) The continuity of the identity map $(\mathcal{C}^{\text{sa}}, d_R) \rightarrow (\mathcal{C}^{\text{sa}}, d_G)$ was observed by Nicolaescu [Nic00, Lemma 1.2].

(2) The continuity of β generalizes [BBF98, Thm. 4.8 and Cor. 4.9] where it is proved that the composition map $\beta \circ \alpha$ is continuous.

PROOF. (1) For $C_1, C_2 \in \mathcal{B}^{\text{sa}}$ we have

$$\begin{aligned} d_W(D + C_1, D + C_2) &= \|(C_1 - C_2)(I + D^2)^{-1/2}\| \\ (2.8) \quad &\leq \|C_1 - C_2\| \|(I + D^2)^{-1/2}\| \\ &\leq d_N(D + C_1, D + C_2), \end{aligned}$$

i.e. $d_W \leq d_N$ and hence α is continuous.

(2) For completeness we briefly recall Nicolaescu's [Nic00] argument to prove the continuity of the identity map $(\mathcal{C}^{\text{sa}}, d_R) \xrightarrow{\text{id}} (\mathcal{C}^{\text{sa}}, d_G)$: for $T \in \mathcal{C}^{\text{sa}}$ we have

$$(2.9) \quad (T + i)^{-1} = (T - i)(I + T^2)^{-1} = (I + T^2)^{-1/2} F(T) - i(I + T^2)^{-1}$$

and

$$(2.10) \quad (I + T^2)^{-1} = I - F(T)^2.$$

Hence, if $F(T_n) \rightarrow F(T)$ then $(I + T_n^2)^{-1} \rightarrow (I + T^2)^{-1}$ and thus also $(I + T_n^2)^{-1/2} \rightarrow (I + T^2)^{-1/2}$. Consequently $(T_n + i)^{-1} \rightarrow (T + i)^{-1}$.

(3) The continuity of β is more complicated. We fix a $T \in \mathcal{B}^{\text{sa}}(W, H)$ and we have to prove the continuity of F at T . Put

$$(2.11) \quad M := \|(T \pm i)^{-1}(I + D^2)^{1/2}\| = \|(I + D^2)^{1/2}(T \pm i)^{-1}\|.$$

Let $0 < q < \frac{1}{2}$ and consider $\tilde{T} \in \mathcal{B}^{\text{sa}}(W, H)$ with $d_W(T, \tilde{T}) \leq \frac{q}{M}$. Then we have

$$(2.12) \quad \|(T - \tilde{T})(T \pm i)^{-1}\|, \|(T \pm i)^{-1}(T - \tilde{T})\| \leq q.$$

The Neumann series then immediately implies

$$(2.13) \quad \|(\tilde{T} + i)^{-1}(T + i)\| \leq \frac{1}{1 - q}.$$

Thus, for $x \in H$ we have

$$(2.14) \quad \|(\tilde{T} + i)^{-1}x\| \leq \frac{1}{1 - q} \|(T + i)^{-1}x\|$$

and

$$(2.15) \quad \|(T + i)^{-1}x\| \leq \|(T + i)^{-1}(\tilde{T} + i)\| \|(\tilde{T} + i)^{-1}x\| \leq (1 + q) \|(\tilde{T} + i)^{-1}x\|.$$

This implies the operator inequalities

$$(2.16) \quad \frac{1}{(1 + q)^2} |T + i|^{-2} \leq |\tilde{T} + i|^{-2} \leq \frac{1}{(1 - q)^2} |T + i|^{-2}.$$

Since the square root is an operator-monotonic increasing function (Kadison and Ringrose [KR97, Prop. 4.2.8]) we may take the square root of these inequalities and after subtracting $|T + i|^{-1}$ we arrive at

$$(2.17) \quad -\frac{q}{1 + q} |T + i|^{-1} \leq |\tilde{T} + i|^{-1} - |T + i|^{-1} \leq \frac{q}{1 - q} |T + i|^{-1}.$$

This gives

$$(2.18) \quad \| |T + i|^{1/2} |\tilde{T} + i|^{-1} |T + i|^{1/2} - I \| \leq \frac{q}{1 - q}.$$

In the following series of estimates we are going to use the estimate Proposition A.1 several times:

$$(2.19) \quad \begin{aligned} & \|F(T) - F(\tilde{T})\| \\ & \leq \| |i + T|^{-1/2} (F(T) - F(\tilde{T})) |i + T|^{1/2} \| \\ & \leq \| |i + T|^{-1/2} (T - \tilde{T}) |i + T|^{-1/2} \| \\ & \quad + \| |T + i|^{-1/2} (\tilde{T}(|i + T|^{-1} - |i + \tilde{T}|^{-1})) |i + T|^{1/2} \| \\ & \leq \| |i + T|^{-1} (T - \tilde{T}) \| \\ & \quad + \| |i + T|^{-1/2} \tilde{T} |i + T|^{-1/2} \| \|I - |i + T|^{1/2} |i + \tilde{T}|^{-1} |i + T|^{1/2}\| \\ & \leq q + \| |i + T|^{-1} \tilde{T} \| \frac{q}{1 - q} \\ & \leq q(1 + \frac{1 + q}{1 - q}). \end{aligned}$$

This shows that if $d_W(T_n, T) \rightarrow 0$ then $F(T_n) \rightarrow F(T)$ and we are done. \square

By a famous example due to Fuglede ([Nic00, Rem. 1.5], [BBLP01, Ex. 2.14]) the Riesz topology on \mathcal{C}^{sa} is strictly stronger than the graph topology. The counterexamples in loc. cit. even have fixed domain, i.e. a sequence of the form $T_n = D + C_n$, $C_n \in \mathcal{B}^{\text{sa}}$, is constructed such that T_n converges in the graph but not in the Riesz topology. We will refine the Fuglede example and show that the four topologies induced by d_N, d_W, d_R, d_G are all different.

Before let us introduce a bit of notation. For metrics d_1, d_2 on a metric space we write $d_1 \succeq d_2$ ($d_1 \succneq d_2$) if the topology induced by d_1 is (strictly) stronger than the one induced by d_2 . Of course, if $d_1 \geq d_2$ then $d_1 \succeq d_2$ but the converse need not be true.

PROPOSITION 2.4. *Let H be a separable complex Hilbert space and let D be a self-adjoint operator in H with compact resolvent.*

- (1) *On $D + \mathcal{B}^{\text{sa}}$ we have $d_N \succeq d_W \succeq d_R \succeq d_G$.*
- (2) *For fixed $R \geq 0$ we have on the space*

$$\{D + C \mid C \in \mathcal{B}^{\text{sa}}, \| |D + i|^{-1} C |D + i| \| \leq R\}$$

that $d_W \succeq d_R$ and $d_R \succeq d_W$, i.e. d_W and d_R induce the same topology on this subset of $D + \mathcal{B}^{\text{sa}}$.

PROOF. It follows from Proposition 2.2 that $d_N \succeq d_W \succeq d_R \succeq d_G$.

Next we prove that on $\{D + C \mid C \in \mathcal{B}^{\text{sa}}, \| |D + i|^{-1} C |D + i| \| \leq R\}$ we also have $d_R \succeq d_W$.

Let $C_n, C \in \mathcal{B}^{\text{sa}}$, $\| |D + i|^{-1} C_n |D + i| \| \leq R$, $\| |D + i|^{-1} C |D + i| \| \leq R$, and assume that $d_R(D + C_n, D + C) \rightarrow 0$, i.e. $F(D + C_n) \rightarrow F(D + C)$, $n \rightarrow \infty$.

We note that it follows from Proposition A.1 that the operators

$$(2.20) \quad |D + i| C_n |D + i|^{-1}, \quad |D + i| C |D + i|^{-1}$$

are bounded (and defined on all of H) and satisfy the same norm bound.

Consider the identity

$$(2.21) \quad \begin{aligned} & F(D + C_n) - F(D + C) \\ &= (D + C_n) [|D + C_n + i|^{-1} - |D + C + i|^{-1}] + (C_n - C) |D + C + i|^{-1}. \end{aligned}$$

We have to show that $\|(C_n - C) |D + i|^{-1}\| \rightarrow 0$. We first note that it suffices to show that $(C_n - C) |D + C + i|^{-1} \rightarrow 0$ strongly. Indeed, if this is the case then for $x \in \mathcal{D}(D)$ we have $(C_n - C)x = ((C_n - C)(D + C + i)^{-1})((D + C + i)x) \rightarrow 0$. Hence $(C_n - C) \rightarrow 0$ strongly on the dense subspace $\mathcal{D}(D)$. Since in view of Proposition A.1 $\|C_n - C\| \leq R$ is uniformly bounded we infer that $(C_n - C) \rightarrow 0$ strongly on H . Now since D has compact resolvent $|D + i|^{-1}$ is compact and since multiplication from the right by compact operators turns strongly convergent sequences into uniformly convergent sequences we indeed conclude that $\|(C_n - C) |D + i|^{-1}\| \rightarrow 0$.

To prove that $(C_n - C) |D + C + i|^{-1} \rightarrow 0$ strongly we assume the contrary. Then there is an $x \in H$ and an $\varepsilon > 0$ such that after possibly considering a subsequence we have

$$(2.22) \quad \|(C_n - C) |D + C + i|^{-1} x\| \geq \varepsilon.$$

Again, since $\|C_n - C\|$ is uniformly bounded and since $\mathcal{D}(D)$ is dense in H we may assume that $x \in \mathcal{D}(D)$.

Since $F(D + C_n) \rightarrow F(D + C)$ (cf. the argument after (2.10)) we have

$$(2.23) \quad x_n := [|D + C_n + i|^{-1} - |D + C + i|^{-1}] x \rightarrow 0, \quad n \rightarrow \infty.$$

Applying Proposition A.2 with $\alpha = 2, \beta = -1$ (and $\alpha = 0, \beta = -1$ and repeatedly using the boundedness of $\| |D + i| C_n |D + i|^{-1} \|, \| |D + i| C |D + i|^{-1} \|$) we infer that

$$(2.24) \quad y_n = (D + C_n)x_n = (D + C_n) [|D + C_n + i|^{-1} - |D + C + i|^{-1}] x$$

is a bounded sequence in $\mathcal{D}(D)$. Since D has compact resolvent the inclusion $\mathcal{D}(D) \hookrightarrow H$ is compact and thus a subsequence of (y_n) converges in H .

Summing up we have proved that there is a subsequence x_{n_k} such that $x_{n_k} \rightarrow 0$ and such that (since C_n is bounded) Dx_{n_k} converges in H . But D is a closed operator, hence $Dx_{n_k} \rightarrow 0$ and thus $y_{n_k} \rightarrow 0$.

Plugging x into the identity (2.21) we arrive at $(C_{n_k} - C)|D + C + i|^{-1}x \rightarrow 0$ contradicting (2.22).

Finally we are going to present three counterexamples which prove the claimed $\not\sim$ relations:

Since D has compact resolvent there is an orthonormal basis $(e_k)_{k=1}^\infty$ of eigenvectors, $De_k = \lambda_k e_k$, and $\lim_{k \rightarrow \infty} |\lambda_k| = \infty$.

(1) Let $C_n \in \mathcal{B}^{\text{sa}}$ be defined by

$$(2.25) \quad C_n e_k := \begin{cases} e_n, & k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then C_n is a self-adjoint rank-one operator, $\|C_n\| = 1$, and hence $d_N(D + C_n, D) = 1$. On the other hand, however, we find

$$(2.26) \quad C_n(I + D^2)^{-1/2} e_k = \begin{cases} (1 + \lambda_n^2)^{-1/2} e_n, & k = n, \\ 0, & \text{otherwise,} \end{cases}$$

and thus

$$(2.27) \quad d_W(D + C_n, D) = (1 + \lambda_n^2)^{-1/2} \xrightarrow{n \rightarrow \infty} 0.$$

This proves $d_N \not\sim d_W$ in part (1) and (2) of the Proposition.

(2) Next we put

$$(2.28) \quad C_n e_k := \begin{cases} \lambda_n e_n, & k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Again, C_n is a self-adjoint rank-one operator, $\|C_n\| = |\lambda_n|$, and

$$(2.29) \quad \begin{aligned} d_W(D + C_n, D) &= \|C_n(I + D^2)^{-1/2}\| \\ &\geq \|C_n(I + D^2)^{-1/2} e_n\| \\ &= \frac{|\lambda_n|}{\sqrt{1 + \lambda_n^2}} \xrightarrow{n \rightarrow \infty} 1. \end{aligned}$$

On the other hand, however, we find

$$(2.30) \quad \|F(D + C_n) - F(D)\| = \left| \frac{2\lambda_n}{\sqrt{1 + (2\lambda_n)^2}} - \frac{\lambda_n}{\sqrt{1 + \lambda_n^2}} \right| \xrightarrow{n \rightarrow \infty} 0.$$

This proves $d_W \not\sim d_R$ on $D + \mathcal{B}^{\text{sa}}$.

(3) The following is the famous example due to Fuglede ([**Nic00**, Rem. 1.5], [**BBLP01**, Ex. 2.14]): put

$$(2.31) \quad C_n e_k := \begin{cases} -2\lambda_n e_n, & k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$(2.32) \quad d_G(D + C_n, D) = \left| (-\lambda_n + i)^{-1} - (\lambda_n + i)^{-1} \right| \rightarrow 0, \quad n \rightarrow \infty.$$

On the other hand, however,

$$(2.33) \quad \begin{aligned} \|F(D + C_n) - F(D)\| \\ \geq \|(F(D + C_n) - F(D))e_n\| &= |2\lambda_n(1 + \lambda_n^2)^{-1/2}| \rightarrow 2, \quad n \rightarrow \infty, \end{aligned}$$

and hence $d_R \not\asymp d_G$ on $D + \mathcal{B}^{\text{sa}}$. \square

REMARK 2.5.

(1) By a result due to Nicolaescu [Nic00, Prop. 1.4] Riesz convergence can also be characterized as follows: let $f : \mathbb{R} \rightarrow \mathbb{C}$ be any continuous function with $f(x) = 1$ for $x \gg 1$ and $f(x) = -1$ for $x \ll -1$. Then a sequence $T_n \in \mathcal{C}^{\text{sa}}$ is d_R -convergent if and only if it is d_G -convergent and $f(T_n)$ is convergent.

In particular this implies that if (T_n) is a sequence of operators with $T_n \geq -C$ for some fixed C then (T_n) is d_R -convergent if and only if it is d_G -convergent.

(2) Proposition 2.4 (2) is sharp in the sense that in general $d_W \not\asymp d_R$ even on $\{D + C \mid C \in \mathcal{B}^{\text{sa}}, \|C\| \leq R\}$. To see this consider

$$(2.34) \quad C_n e_k := \begin{cases} e_n, & k = 1, \\ e_1, & k = n, \\ 0, & \text{otherwise.} \end{cases}$$

C_n is a self-adjoint rank-two operator, $\|C_n\| = 1$. We have

$$(2.35) \quad \|(D + C_n + i)^{-1}(D + i)\| \leq 1 + \|C_n\| = 2$$

and thus

$$(2.36) \quad \begin{aligned} d_G(D + C_n, D) &= \|(D + C_n + i)^{-1} - (D + i)^{-1}\| \\ &= \|(D + C_n + i)^{-1}C_n(D + i)^{-1}\| \\ &\leq 2\|(D + i)^{-1}C_n(D + i)^{-1}\|. \end{aligned}$$

Furthermore,

$$(2.37) \quad (D + i)^{-1}C_n(D + i)^{-1}e_k = \begin{cases} \frac{1}{(\lambda_1 + i)(\lambda_n + i)}e_n, & k = 1, \\ \frac{1}{(\lambda_1 + i)(\lambda_n + i)}e_1, & k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently

$$(2.38) \quad d_G(D + C_n, D) \leq \frac{2}{\sqrt{1 + \lambda_n^2}} \xrightarrow{n \rightarrow \infty} 0.$$

With a little more effort one can show that also $d_R(D + C_n, D) \rightarrow 0$. However, if e.g. D is essentially positive then $d_R(D + C_n, D) \rightarrow 0$ follows already from the previous remark.

On the other hand, however,

$$(2.39) \quad d_W(D + C_n, D) = \|C_n(I + D^2)^{-1/2}\| \geq \|C_n(I + D^2)^{-1/2}e_1\| = \frac{1}{\sqrt{1 + \lambda_1^2}},$$

and thus $D + C_n$ does not converge to D in the d_W -metric.

In view of Proposition 2.4 (2) this means that $\|(D + i)^{-1}C_n(D + i)\|$ must be unbounded. Indeed,

$$(2.40) \quad \begin{aligned} \|(D + i)^{-1}C_n(D + i)\| &\geq \|(D + i)^{-1}C_n(D + i)e_n\| \\ &= |\lambda_n + i||\lambda_1 + i|^{-1} \rightarrow \infty, \quad n \rightarrow \infty. \end{aligned}$$

(3) We leave it as an intriguing open problem to find out whether the metrics d_R and d_G induce equivalent topologies on $\{D + C \mid C \in \mathcal{B}^{\text{sa}}, \|C\| \leq R\}$.

3. The spectral flow, index of a pair of projections, and the abstract Toeplitz index theorem

In this section we relate the spectral flow of certain Riesz continuous paths to the index of the positive spectral projections at the endpoints. This generalizes the work of Bunke [Bun94] who has considered the special case of families of the form $D_t := D + tR$ where D is a Dirac operator on a compact manifold and R is a self-adjoint bundle endomorphism satisfying additional assumptions.

DEFINITION 3.1. Let P, Q be orthogonal projections in the Hilbert space H . The pair (P, Q) is called a Fredholm pair if the map $Q : \text{im } P \rightarrow \text{im } Q$ is a Fredholm operator. The index of this operator is denoted by $\text{ind}(P, Q)$.

As pointed out by the referee the notion of the index of a pair of projections was introduced by Brown, Douglas, and Fillmore [BDF73] who called it the "essential codimension". Booß-Bavnbek and Wojciechowski [BBW93, p. 129 ff] used the terminology "virtual codimension".

Avron, Seiler, and Simon [ASS94] gave a systematic account of Fredholm pairs. In particular they showed that a pair (P, Q) of orthogonal projections is Fredholm if and only if $\pm 1 \notin \text{spec}_{\text{ess}}(P - Q)$ [ASS94, Prop. 3.1]. The latter means that the images $\pi(P), \pi(Q)$ in the Calkin algebra \mathcal{B}/\mathcal{K} satisfy

$$(3.1) \quad \|\pi(P) - \pi(Q)\| < 1.$$

In [Phi97] it was shown that the index of a pair of projections can be developed solely from this inequality and that it generalizes to arbitrary semifinite von Neumann factors. Furthermore, this was used to give a completely general definition of spectral flow for continuous paths in the bounded case and hence in the Riesz metric as well.

For the basic properties of Fredholm pairs we refer to [ASS94] and [BL01] whose results we use freely. We only record the following which is proved in [Bun94, Lemma 2.4] only in a special case.

LEMMA 3.2. *Let $(P(t), Q(t)), 0 \leq t \leq 1$, be a norm continuous path of Fredholm pairs. Then $\text{ind}(P(0), Q(0)) = \text{ind}(P(1), Q(1))$.*

PROOF. The proof follows the one in [Bun94, Lemma 2.4]. As in loc. cit. we emphasize that the result is not standard since domain and range of $Q(t) : \text{im } P(t) \rightarrow \text{im } Q(t)$ varies with t .

By a standard fact often used in operator K-theory (Blackadar [Bla86, Prop. 4.3.3]) there exist continuous families of unitaries $U, V : [0, 1] \rightarrow \mathcal{U}$, $U(0) = V(0) = I$ such that $P(t) = U(t)P(0)U(t)^*$ and $Q(t) = V(t)Q(0)V(t)^*$. Hence

$$(3.2) \quad \text{ind}(P(t), Q(t)) = \text{ind}\left(Q(0)V(t)^*U(t)P(0) : \text{im}(P(0)) \rightarrow \text{im}(Q(0))\right).$$

Now $Q(0)V(t)^*U(t)P(0)$ is a norm-continuous family of Fredholm operators between fixed Hilbert spaces. Thus the index does not depend on t as claimed. \square

LEMMA 3.3. *Let $f : [0, 1] \rightarrow (\mathcal{CF}^{\text{sa}}, d_R)$ be a Riesz continuous path of self-adjoint Fredholm operators. Furthermore, assume that $\lambda \notin \text{spec } f(t)$ for all t . Then the path of spectral projections $t \mapsto 1_{(\lambda, \infty)}(f(t))$ is norm-continuous.*

In view of the Fuglede example (see (3) in the proof of Proposition 2.4) we cannot expect this to hold for graph continuous paths.

PROOF. We first note that the Riesz transform of f , $F(f(t))$, is a norm-continuous path of bounded self-adjoint Fredholm operators and by the Spectral Theorem we have

$$(3.3) \quad 1_{(\lambda, \infty)}(f(t)) = 1_{(F(\lambda), \infty)}(F(f(t))).$$

Hence we are reduced to the case of a norm-continuous family of bounded operators for which the claim is (fairly) clear in view of the functional calculus. Namely, since

$$(3.4) \quad \text{spec}(F(f(t))) \subset [-1, 1]$$

we have

$$(3.5) \quad 1_{(F(\lambda), \infty)}(F(f(t))) = \frac{1}{2\pi i} \oint_{|z - (F(\lambda) + 2)| = 2} (z - F(f(t)))^{-1} dz.$$

Now the right hand side of (3.5) depends continuously on t . \square

Let $T \in \mathcal{C}^{\text{sa}}$. We recall that a symmetric operator S with $\mathcal{D}(S) \supset \mathcal{D}(T)$ is called T -compact (Kato [Kat76, Sec. IV.1.3]) if $S(T+i)^{-1}$ is a compact operator. Note that in this case $T+S$ with domain $\mathcal{D}(T)$ is a self-adjoint operator, too.

PROPOSITION 3.4. *Let $T \in \mathcal{C}^{\text{sa}}$ and let S be a T -compact symmetric operator in H . Then the difference of the Riesz transforms,*

$$F(T+S) - F(T),$$

is compact.

The proof of this intuitively clear result is more complicated than expected.

PROOF. (1) We first deal with a special case: assume for the moment that S is bounded, compact and $\text{im } S \subset \mathcal{D}(T)$. Hence $R := (T+S)^2 - T^2 = TS + S(T+S)$ is defined on $\mathcal{D}(T) = \mathcal{D}(T+S)$. We have

$$(3.6) \quad F(T+S) - F(T) = (T+S)(|T+S+i|^{-1} - |T+i|^{-1}) + S|T+i|^{-1}.$$

In view of the assumptions on S the operators $S(|T+S+i|^{-1} - |T+i|^{-1})$ and $S|T+i|^{-1}$ are compact. It remains to prove that $T(|T+S+i|^{-1} - |T+i|^{-1})$ is compact. Using the resolvent equation we find (cf. Remark A.3 and (A.12))

$$(3.7) \quad \begin{aligned} & T(|T+S+i|^{-1} - |T+i|^{-1}) \\ &= -\frac{2}{\pi} \int_0^\infty T(I + T^2 + x^2)^{-1} (TS + S(T+S))(I + (T+S)^2 + x^2)^{-1} dx \end{aligned}$$

Now the Spectral Theorem gives the estimates

$$(3.8) \quad \|T^r(I + T^2 + x^2)^{-1} T^s\| = O(x^{-2+r+s}), \quad x \rightarrow \infty, \quad r, s \in \{0, 1\},$$

and similarly for $T+S$ in place of T . This shows that the integrand in (3.7) is a continuous function with values in the compact operators which is $O(x^{-2})$ as $x \rightarrow \infty$. Hence the integral in (3.7) is a compact operator, too.

(2) Treating the general case we introduce the spectral projections $P_n := 1_{[-n, n]}(T)$ and put $S_n := P_n S P_n$. Since P_n maps H continuously into $\mathcal{D}(T)$ we find that S_n is a bounded compact operator with $\text{im}(S_n) \subset \mathcal{D}(T)$, hence the situation (1) applies to S_n . Now consider

$$(3.9) \quad (S_n - S)(T+i)^{-1} = (P_n - I)S(T+i)^{-1}P_n + S(T+i)^{-1}(P_n - I).$$

P_n converges to I strongly and is norm bounded by 1. Since $S(T+i)^{-1}$ is compact we find that $(S_n - S)(T+i)^{-1}$ converges in the norm to 0. In other words $T + S_n$ converges to $T + S$ in the $d_{\mathcal{D}(T)}$ -metric. Then, in view of Proposition 2.2, $T + S_n$ converges to $T + S$ also in the Riesz metric. Consequently $F(T + S_n) - F(T)$ converges to $F(T + S) - F(T)$.

$F(T + S_n) - F(T)$ is compact by the proved case (1) and thus $F(T + S) - F(T)$ is compact, too. \square

COROLLARY 3.5. *Under the assumptions of the previous Proposition let $\lambda \notin \text{spec}_{\text{ess}} T$. Then the difference of the spectral projections $1_{[\lambda, \infty)}(T + S) - 1_{[\lambda, \infty)}(T)$ is a compact operator.*

PROOF. In light of the previous Proposition it suffices to prove the claim for bounded T and compact S . Otherwise replace T by $F(T)$ and S by $F(T + S) - F(T)$.

Since S is compact we have $\text{spec}_{\text{ess}}(T) = \text{spec}_{\text{ess}}(T + S)$. Hence λ is at most an isolated eigenvalue of finite multiplicity of T or $T + S$. Thus we may choose $\mu < \lambda$ such that $\text{spec}(T) \cap (\mu, \lambda) = \text{spec}(T + S) \cap (\mu, \lambda) = \emptyset$. Then

$$(3.10) \quad 1_{[\lambda, \infty)}(T) = 1_{(\mu, \infty)}(T), \quad 1_{[\lambda, \infty)}(T + S) = 1_{(\mu, \infty)}(T + S).$$

Now choose $a > \mu$ such that $\sup(\text{spec}(T) \cup \text{spec}(T + S)) < 2a - \mu$. Then

$$(3.11) \quad \begin{aligned} & 1_{[\lambda, \infty)}(T + S) - 1_{[\lambda, \infty)}(T) \\ &= \frac{1}{2\pi i} \oint_{|z-a|=a-\mu} (z - T - S)^{-1} - (z - T)^{-1} dz \\ &= \frac{1}{2\pi i} \oint_{|z-a|=a-\mu} (z - T)^{-1} S (z - T - S)^{-1} dz, \end{aligned}$$

and this is compact since S is compact. \square

THEOREM 3.6. *Let $[0, 1] \ni t \mapsto T_t \in (\mathcal{CF}^{\text{sa}}, d_R)$ be a Riesz continuous path of self-adjoint Fredholm operators. Assume furthermore that the domain of T_t does not depend on t , $\mathcal{D}(T_t) = \mathcal{D}(T_0)$, and that for $t \in [0, 1]$ the difference $T_t - T_0$ is T_0 -compact. Then the pair $(1_{[0, \infty)}(T_1), 1_{[0, \infty)}(T_0))$ is Fredholm and*

$$\text{SF}(T_t)_{t \in [0, 1]} = \text{ind}(1_{[0, \infty)}(T_1), 1_{[0, \infty)}(T_0)).$$

We single out a special case which will be of interest in the proof of the uniqueness of the spectral flow.

COROLLARY 3.7. *Let $(T_t)_{t \in [0, 1]}$ be a continuous path of self-adjoint complex $n \times n$ matrices. Then*

$$\text{SF}(T_t)_{t \in [0, 1]} = \text{rank}(1_{[0, \infty)}(T_1)) - \text{rank}(1_{[0, \infty)}(T_0)).$$

PROOF. For orthogonal projections P, Q in a finite-dimensional Hilbert space we clearly have $\text{ind}(P, Q) = \text{rank } P - \text{rank } Q$. \square

PROOF OF THEOREM 3.6. We may choose a subdivision $0 = t_0 < t_1 < \dots < t_n = 1$ such that there exist $\varepsilon_j > 0$ with $\pm \varepsilon_j \notin \text{spec}(T_t)$ and

$$(3.12) \quad \text{spec}_{\text{ess}}(T_t) \cap [-\varepsilon_j, \varepsilon_j] = \emptyset, \quad t_{j-1} \leq t \leq t_j, \quad j = 1, \dots, n.$$

Then we have by Definition 1.1

$$(3.13) \quad \text{SF}\left((T_t)_{t \in [0, 1]}\right) = \sum_{j=1}^n \left(\text{rank}(1_{[0, \varepsilon_j)}(T_{t_j})) - \text{rank}(1_{[0, \varepsilon_j)}(T_{t_{j-1}})) \right).$$

In view of Lemma 3.3 we may, after refining the subdivision, assume that for $t, t' \in [t_{j-1}, t_j]$ we have

$$(3.14) \quad \|1_{(\varepsilon_j, \infty)}(T_{t'}) - 1_{(\varepsilon_j, \infty)}(T_t)\| < 1.$$

Then $1_{(\varepsilon_j, \infty)}(T_{t'})$ maps $\text{im } 1_{(\varepsilon_j, \infty)}(T_t)$ bijectively onto $\text{im } 1_{(\varepsilon_j, \infty)}(T_{t'})$. Hence

$$(3.15) \quad \begin{aligned} \text{ind}(1_{(\varepsilon_j, \infty)}(T_{t_{j-1}}) : \text{im } 1_{[0, \infty)}(T_{t_j}) \rightarrow \text{im } 1_{[0, \infty)}(T_{t_{j-1}})) \\ = \text{rank}(1_{[0, \varepsilon_j)}(T_{t_j})) - \text{rank}(1_{[0, \varepsilon_j)}(T_{t_{j-1}})). \end{aligned}$$

Furthermore, since $1_{[\varepsilon_j, \infty)}(T_{t_{j-1}}) - 1_{[0, \infty)}(T_{t_{j-1}})$ is of finite rank we find

$$(3.16) \quad \begin{aligned} \text{ind}(1_{[0, \infty)}(T_{t_j}), 1_{[0, \infty)}(T_{t_{j-1}})) \\ = \text{rank}(1_{[0, \varepsilon_j)}(T_{t_j})) - \text{rank}(1_{[0, \varepsilon_j)}(T_{t_{j-1}})). \end{aligned}$$

Equations (3.13) and (3.16) give

$$(3.17) \quad \text{SF}\left((T_t)_{t \in [0, 1]}\right) = \sum_{j=1}^n \text{ind}(1_{[0, \infty)}(T_{t_j}), 1_{[0, \infty)}(T_{t_{j-1}})).$$

So far we have not used the assumption that the domain of T_t is independent of t and the difference $T_t - T_0$ is T_0 -compact. Hence (3.17) holds for any Riesz continuous path in \mathcal{CF}^{sa} .

Now in view of our compactness assumption Corollary 3.5 implies that the differences $1_{[0, \infty)}(T_{t_j}) - 1_{[0, \infty)}(T_{t_{j-1}})$ are compact. In particular the difference $1_{[0, \infty)}(T_1) - 1_{[0, \infty)}(T_0)$ is compact and hence $(1_{[0, \infty)}(T_1), 1_{[0, \infty)}(T_0))$ is a Fredholm pair [ASS94, Thm. 3.4].

Since the index of a pair of projections satisfies $\text{ind}(P, R) = \text{ind}(P, Q) + \text{ind}(Q, R)$ if $P - Q$ or $Q - R$ is compact [ASS94, Thm. 3.4] the right hand side of (3.17) indeed equals $\text{ind}(1_{[0, \infty)}(T_1), 1_{[0, \infty)}(T_0))$ and the Theorem is proved. \square

We record explicitly that, as noted in the proof, equation (3.17) holds for any Riesz continuous path. For norm continuous paths of bounded self-adjoint Fredholm operators this was already shown in [Phi97].

COROLLARY 3.8. *Let $[0, 1] \ni t \mapsto T_t \in (\mathcal{CF}^{\text{sa}}, d_R)$ be a Riesz continuous path of self-adjoint Fredholm operators. Choose a subdivision as in the beginning of the proof of Theorem 3.6 which is fine enough such that for $t, t' \in [t_{j-1}, t_j]$ we have*

$$(3.18) \quad \|1_{(\varepsilon_j, \infty)}(T_{t'}) - 1_{(\varepsilon_j, \infty)}(T_t)\| < 1.$$

Then we have

$$(3.19) \quad \text{SF}\left((T_t)_{t \in [0, 1]}\right) = \sum_{j=1}^n \text{ind}(1_{[0, \infty)}(T_{t_j}), 1_{[0, \infty)}(T_{t_{j-1}})).$$

As a consequence of Theorem 3.6 we note the abstract Toeplitz Index Theorem (cf. [Bun94, Prop. 3.1]).

PROPOSITION 3.9. *Let $D \in \mathcal{CF}^{\text{sa}}$ and let $W \in \mathcal{U}$ be a unitary operator with $W^*(\mathcal{D}(D)) = \mathcal{D}(D)$ and $[D, W]$ D -compact. Let $P_+ := 1_{[0, \infty)}(D)$. Then the Toeplitz operator $P_+ W P_+ : \text{im } P_+ \rightarrow \text{im } P_+$ is Fredholm and*

$$\text{ind}(P_+ W P_+) = \text{SF}\left((1-s)D + sW D W^*\right)_{0 \leq s \leq 1}.$$

REMARK 3.10. Note that P_+WP_+ is Fredholm on $\text{im } P_+$ if and only if

$$(3.20) \quad P_+WP_+ + (I - P_+) = I + (W - I)P_+ - [W, P_+]P_+$$

is Fredholm on H with the same index. Since $[W, P_+]$ is compact we conclude

$$(3.21) \quad \text{ind}(P_+WP_+) = \text{ind}(I + (W - I)P_+)$$

which gives [Bun94, Prop. 3.1].

To see the compactness of $[W, P_+]$ let $P_+(WDW^*) := 1_{[0, \infty)}(WDW^*) = WP_+W^*$. Then

$$(3.22) \quad [W, P_+]W^* = WP_+W^* - P_+ = P_+(WDW^*) - P_+.$$

Since $WDW^* - D = [W, D]W^*$ is D -compact the operator $P_+(WDW^*) - P_+$ is compact in view of Corollary 3.5.

PROOF OF PROPOSITION 3.9. By assumption we have

$$(3.23) \quad D_s := (1 - s)D + sWDW^* = D + s[W, D]W^*.$$

Hence we may apply Theorem 3.6 and find that (WP_+W^*, P_+) is a Fredholm pair and

$$(3.24) \quad \begin{aligned} \text{SF}(D_s) &= \text{ind}(WP_+W^*, P_+) \\ &= \text{ind}(P_+ : \text{im } WP_+W^* \rightarrow \text{im } P_+) \\ &= \text{ind}(P_+WP_+ : \text{im } P_+ \rightarrow \text{im } P_+) \end{aligned}$$

and the result is proved. \square

4. The Theorem of Cordes–Labrousse revisited

PROPOSITION 4.1. \mathcal{B}^{sa} is open and dense in $(\mathcal{CF}^{\text{sa}}, d_G)$ and also in $(\mathcal{CF}^{\text{sa}}, d_R)$. Moreover, the topology induced by the graph resp. Riesz metric on \mathcal{B}^{sa} coincides with the norm topology.

That the graph metric induces the norm topology on bounded operators is due to Cordes and Labrousse [CL63, Addendum] who also observed that the bounded operators are open in the graph metric. That \mathcal{B}^{sa} is dense in $(\mathcal{CF}^{\text{sa}}, d_G)$ was observed in [BBLP01, Prop. 1.6].

PROOF. By Proposition 2.2 we know that the Riesz metric is stronger than the graph metric and applying Proposition 2.2 with $D = 0$ (which is not excluded!) we see that the natural inclusion $\mathcal{B}^{\text{sa}} \hookrightarrow (\mathcal{CF}^{\text{sa}}, d_R)$ is continuous. Hence it suffices to show that \mathcal{B}^{sa} is open in the graph topology, that \mathcal{B}^{sa} is dense in $(\mathcal{CF}^{\text{sa}}, d_R)$ and that the topology induced by the graph metric on \mathcal{B}^{sa} coincides with the norm topology.

(1) Fix $R > 0$ and $T \in \mathcal{B}^{\text{sa}}$, $\|T\| \leq R$. Consider $\tilde{T} \in \mathcal{CF}^{\text{sa}}$ with $d_G(T, \tilde{T}) < \frac{1}{2}(1 + R)^{-1}$. Then

$$(4.1) \quad (\tilde{T} + i)^{-1} = (T + i)^{-1} \left(I - (T + i)((T + i)^{-1} - (\tilde{T} + i)^{-1}) \right)$$

is invertible with bounded inverse since

$$(4.2) \quad \left\| (T + i) \left((T + i)^{-1} - (\tilde{T} + i)^{-1} \right) \right\| \leq (1 + R)d_G(T, \tilde{T}) < \frac{1}{2},$$

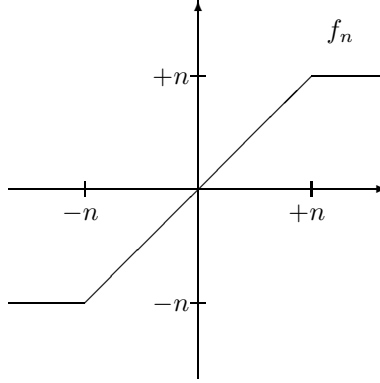


FIGURE 1. Function f_n . $f_n(T)$ is bounded and converges in the graph distance to the operator T .

and

$$(4.3) \quad (\tilde{T} + i) = \sum_{n=0}^{\infty} \left((T + i)((T + i)^{-1} - (\tilde{T} + i)^{-1}) \right)^n (T + i).$$

Hence the ball $B_{d_G}(T, \frac{1}{2}(1 + R)^{-1})$ is contained in \mathcal{B}^{sa} which proves that \mathcal{B}^{sa} is open in $(\mathcal{C}^{\text{sa}}, d_G)$.

To prove that it is dense even with respect to the Riesz metric, we consider $T \in \mathcal{C}^{\text{sa}}$ and denote by $(E_\lambda)_{\lambda \in \mathbb{R}}$ the spectral resolution of T . Let f_n be the function sketched in Figure 1.

We put

$$(4.4) \quad f_n(T) = \int_{[-n, n]} \lambda dE_\lambda + \int_{|\lambda| > n} n(\text{sgn } \lambda) dE_\lambda \in \mathcal{B}^{\text{sa}}$$

and find

$$(4.5) \quad \begin{aligned} & \|F(T) - F(f_n(T))\| \\ &= \left\| \int_{|\lambda| > n} \frac{\lambda}{\sqrt{1 + \lambda^2}} - \frac{n(\text{sgn } \lambda)}{\sqrt{1 + n^2}} dE_\lambda \right\| \\ &\leq \sup_{|\lambda| > n} \left| \frac{\lambda}{\sqrt{1 + \lambda^2}} - \frac{n(\text{sgn } \lambda)}{\sqrt{1 + n^2}} \right| \\ &\leq \left| \frac{n}{\sqrt{1 + n^2}} - 1 \right|, \end{aligned}$$

hence $\lim_{n \rightarrow \infty} d_R(T, f_n(T)) = 0$.

(2) Let $T \in \mathcal{B}^{\text{sa}}$, $\|T\| \leq R$. Then, for $\tilde{T} \in \mathcal{B}^{\text{sa}}$ with $d_G(T, \tilde{T}) < \frac{1}{2}(1 + R)^{-1}$ we have in view of (4.3)

$$(4.6) \quad \|T - \tilde{T}\| \leq \sum_{n=1}^{\infty} (1 + R)^{n+1} d_G(T, \tilde{T})^n \leq 2(1 + R)^2 d_G(T, \tilde{T}).$$

Conversely, if $\|T - \tilde{T}\| < \frac{1}{2}$ we find

$$(4.7) \quad \begin{aligned} (\tilde{T} + i)^{-1} &= (I - (T + i)^{-1}(T - \tilde{T}))^{-1}(T + i)^{-1} \\ &= \sum_{n=0}^{\infty} ((T + i)^{-1}(T - \tilde{T}))^n (T + i)^{-1} \end{aligned}$$

and hence

$$(4.8) \quad d_G(T, \tilde{T}) \leq \sum_{n=1}^{\infty} \|T - \tilde{T}\|^n \leq 2\|T - \tilde{T}\|.$$

(4.6) and (4.8) show that the topologies induced by d_G and $\|\cdot\|$ on \mathcal{B}^{sa} coincide. \square

(4.6) and (4.8) show a bit more. Namely, given $T_0 \in \mathcal{B}^{\text{sa}}$, $R := \|T_0\| + 1$ put $r := \frac{1}{4}(1 + R)^{-2}$. Then the ball $B_{d_G}(T_0, r)$ is open in the graph and the norm topology. For $T \in B_{d_G}(T_0, r)$ we find in view of (4.6)

$$(4.9) \quad \|T - T_0\| \leq 2(1 + R)^2 r = \frac{1}{2}$$

thus $\|T\| \leq R$. Hence (4.6) and (4.8) may be applied to arbitrary $T, \tilde{T} \in B_{d_G}(T_0, r)$ and we find

$$(4.10) \quad \frac{1}{2}d_G(T, \tilde{T}) \leq \|T - \tilde{T}\| \leq 2(1 + R)^2 d_G(T, \tilde{T}).$$

Hence the norm distance and the d_G -distance are equivalent on $B_{d_G}(T_0, r)$.

Still, the norm distance and the d_G -distance are not globally equivalent! The reason is that \mathcal{B}^{sa} is norm complete and at the same time d_G -dense in \mathcal{C}^{sa} .

4.1. The stability of the index. We are going to present a concise proof of the Theorem of Cordes–Labrousse on the stability of the index in a very general context.

Let $H := H^+ \oplus H^-$ be a \mathbb{Z}_2 -graded Hilbert space with grading operator

$$(4.11) \quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We treat the G -equivariant index and the Clifford index simultaneously:

Case I. Let G be a compact Lie group and let $\varrho : G \rightarrow \mathcal{U}^{\text{ev}}$ be a unitary representation of G into the space of even operators on H , i.e. $\varrho(g)\alpha = \alpha\varrho(g)$ for $g \in G$.

Spaces of *odd* G -equivariant operators are denoted by a subscript G , e.g. $\mathcal{CF}_G^{\text{sa}}$, etc. Here, an operator T is called G -equivariant if it commutes with $\varrho(g)$, $g \in G$.

Case II. Denote by $C\ell_n$ the real Clifford algebra (Lawson and Michelsohn [LM89, Chap. I]), i.e. $C\ell_n$ is the universal real C^* -algebra generated by unitaries e_1, \dots, e_n subject to the relations

$$(4.12) \quad e_i e_j + e_j e_i = -2\delta_{ij}.$$

$C\ell_n$ is \mathbb{Z}_2 -graded with the generators e_j being of odd degree.

Let $\varrho : C\ell_n \rightarrow \mathcal{B}$ be a faithful unital graded $*$ -representation of $C\ell_n$ on \mathcal{B} .

Spaces of *odd* $C\ell_n$ -invariant operators are denoted by a subscript n , for example $\mathcal{CF}_n^{\text{sa}}$, etc.

In both cases we now consider an *odd* ϱ -equivariant self-adjoint Fredholm operator $T \in \mathcal{CF}_G^{\text{sa}}(\mathcal{CF}_n^{\text{sa}})$. Then $\ker T$ is a \mathbb{Z}_2 -graded ϱ -module.

Case I. Denote by $(\ker T)^\pm := \ker T \cap \ker(\alpha \mp I)$ the \pm -part of $\ker T$. Then $(\ker T)^\pm$ are G -modules. One puts

$$(4.13) \quad \text{ind}_G(T) := [\ker T] := [(\ker T)^+] - [(\ker T)^-] \in R(G),$$

where $R(G)$ is the ring of virtual finite-dimensional representations of G , i.e. it is the Grothendieck group of the semiring of equivalence classes of finite-dimensional representations.

Case II. Following Atiyah, Bott, and Shapiro [ABS64] (cf. also [LM89, Sec. I.9]) let $\hat{\mathcal{M}}_n$ be the Grothendieck group of equivalence classes of finite-dimensional \mathbb{Z}_2 -graded $C\ell_n$ -modules. Then there is a canonical isomorphism

$$(4.14) \quad \hat{\mathcal{M}}_n / \hat{\mathcal{M}}_{n+1} \simeq KO^{-n}(pt).$$

Recall that $KO^{-n}(pt)$ is 8-periodic, $KO^0(pt) \simeq KO^{-4}(pt) \simeq \mathbb{Z}$, $KO^{-1}(pt) \simeq KO^{-2}(pt) \simeq \mathbb{Z}_2$, and the remaining groups vanish.

The isomorphism

$$(4.15) \quad \hat{\mathcal{M}}_0 / \hat{\mathcal{M}}_1 \simeq KO^0(pt) \simeq \mathbb{Z}$$

is given by sending the graded vector space V (a $C\ell_0$ -module) to its graded dimension $\dim_{\mathbb{Z}_2} V := \dim V^+ - \dim V^-$.

Again, $\ker T$ is a \mathbb{Z}_2 -graded $C\ell_n$ -module and one puts

$$(4.16) \quad \text{ind}_n T := [\ker T] \in \hat{\mathcal{M}}_n / \hat{\mathcal{M}}_{n+1}.$$

Note that an odd self-adjoint Fredholm operator T takes the form

$$(4.17) \quad T = \begin{pmatrix} 0 & (T^+)^* \\ T^+ & 0 \end{pmatrix}$$

and in view of (4.15) $\text{ind}_0 T$ is nothing but the ordinary Fredholm index of T^+ .

LEMMA 4.2. *Let $T \in \mathcal{CF}_G^{\text{sa}}$ or $T \in \mathcal{CF}_n^{\text{sa}}$. Let $\varepsilon > 0$ such that $\pm\varepsilon \notin \text{spec } T$ and $[-\varepsilon, \varepsilon] \cap \text{spec}_{\text{ess}}(T) = \emptyset$. Then $\text{im } 1_{[-\varepsilon, \varepsilon]}(T)$ is a G -module ($C\ell_n$ -module) and*

$$[\text{im}(1_{[-\varepsilon, \varepsilon]}(T))] = [\ker T]$$

in $R(G)$ resp. $\hat{\mathcal{M}}_n / \hat{\mathcal{M}}_{n+1}$.

PROOF. We first note that the choice of $\varepsilon > 0$ is possible. Namely, since T is a Fredholm operator 0 is not in the essential spectrum. Hence, 0 is at most an isolated point of $\text{spec } T$. Since spectrum and essential spectrum are closed one may choose $\varepsilon > 0$ as stated.

Abbreviate $V := \text{im}(1_{[-\varepsilon, \varepsilon]}(T))$. Then we have a ϱ -equivariant decomposition

$$(4.18) \quad V = \ker T \bigoplus_{0 < \lambda < \varepsilon} \ker(T^2 - \lambda^2).$$

Now consider $\lambda > 0$:

Case I. $|T|^{-1}T : \ker(T^2 - \lambda^2)^\pm \rightarrow \ker(T^2 - \lambda^2)^\mp$ is a G -equivariant isomorphism, hence $[\ker(T^2 - \lambda^2)] = 0$ in $R(G)$.

Case II. On $\ker(T^2 - \lambda^2)$ consider $J := |T|^{-1}T$. J is odd, unitary, and $J^2 = I$. Thus with respect to the grading it takes the form

$$(4.19) \quad \begin{pmatrix} 0 & (J^+)^* \\ J^+ & 0 \end{pmatrix}.$$

Put

$$(4.20) \quad E_{n+1} := \begin{pmatrix} 0 & (-J^+)^* \\ J^+ & 0 \end{pmatrix}.$$

Again, E_{n+1} is odd, unitary, and $E_{n+1}^2 = -I$. Moreover, E_{n+1} anticommutes with $\varrho(e_k)$, $k = 1, \dots, n$. Hence $\varrho(e_1), \dots, \varrho(e_n), E_{n+1}$ make $\ker(T^2 - \lambda^2)$ into a graded Cl_{n+1} -module and hence $[\ker(T^2 - \lambda^2)] = 0$ in $\hat{\mathcal{M}}_n / \hat{\mathcal{M}}_{n+1}$.

In view of (4.18) we thus have in both cases $[\mathrm{im}(1_{[-\varepsilon, \varepsilon]}(T))] = [\ker T]$. \square

For the ordinary Fredholm index the following Theorem is due to Cordes and Labrousse [CL63]. For bounded operators in the current equivariant context it can be found in [LM89, Sec. III.10]. However, since the Riesz topology and the graph topology are different (Proposition 2.4), (10.8) in [LM89] is problematic and valid only in the context of unbounded operators with a fixed domain.

THEOREM 4.3. *The G -index*

$$\mathrm{ind}_G : \mathcal{CF}_G^{\mathrm{sa}} \rightarrow R(G)$$

and the Clifford index

$$\mathrm{ind}_n : \mathcal{CF}_n^{\mathrm{sa}} \rightarrow KO^{-n}(pt)$$

are locally constant with respect to the graph topology on $\mathcal{CF}^{\mathrm{sa}}$.

PROOF. Fix a $T \in \mathcal{CF}_G^{\mathrm{sa}}$ (resp. $T \in \mathcal{CF}_n^{\mathrm{sa}}$). Since T is a Fredholm operator, $0 \notin \mathrm{spec}_{\mathrm{ess}}(T)$. Thus there is an $\varepsilon > 0$ such that $\pm\varepsilon \notin \mathrm{spec} T$ and $[-\varepsilon, \varepsilon] \cap \mathrm{spec}_{\mathrm{ess}}(T) = \emptyset$.

Moreover, in view of [BBLP01, Prop. 2.10] there is an open neighborhood $\mathcal{N} \subset \mathcal{CF}_G^{\mathrm{sa}}$ ($\mathcal{CF}_n^{\mathrm{sa}}$) of T such that $\mathcal{N} \ni S \mapsto 1_{[-\varepsilon, \varepsilon]}(S)$ is continuous and finite-rank projection valued.

In particular, making \mathcal{N} smaller if necessary, we may assume that

$$(4.21) \quad \|1_{[-\varepsilon, \varepsilon]}(S) - 1_{[-\varepsilon, \varepsilon]}(T)\| < 1$$

for all $S \in \mathcal{N}$.

It is well-known that if two orthogonal projections P, Q satisfy $\|P - Q\| < 1$ then P maps $\mathrm{im} Q$ isomorphically onto $\mathrm{im} P$. Hence $1_{[-\varepsilon, \varepsilon]}(S)$ is a ϱ -equivariant isomorphism from $\mathrm{im}(1_{[-\varepsilon, \varepsilon]}(T))$ onto $\mathrm{im}(1_{[-\varepsilon, \varepsilon]}(S))$. Thus by Lemma 4.2

$$(4.22) \quad [\ker S] = [\mathrm{im}(1_{[-\varepsilon, \varepsilon]}(S))] = [\mathrm{im}(1_{[-\varepsilon, \varepsilon]}(T))] = [\ker T]$$

for $S \in \mathcal{N}$ and the Theorem is proved. \square

5. Uniqueness of the spectral flow

5.1. The general set-up. We start fixing some basic notation und introducing the problem:

DEFINITION 5.1. For a topological space X and a subspace $Y \subset X$ we denote by $\Omega(X, Y)$ the set of paths $f : [0, 1] \rightarrow X$ with endpoints in Y . Instead of $\Omega(X, X)$ we also write $\Omega(X)$. Paths are always assumed to be continuous.

Paths $f, g \in \Omega(X, Y)$ are (free) homotopic if there is a continuous map $H : [0, 1] \times [0, 1] \rightarrow X$ with the properties

- (1) $H(0, \cdot) = f, H(1, \cdot) = g$,
- (2) $H(s, 0) \in Y, H(s, 1) \in Y$ for all s .

The set of homotopy classes in this sense is denoted by $\tilde{\pi}_1(X, Y)$.

Note that we do not require a base point in Y to stay fixed during the deformation, however endpoints are only allowed to move within Y . Therefore $\tilde{\pi}_1(X, Y)$ is not the relative homotopy set usually introduced in algebraic topology textbooks. If y_0 is a base point in Y then the relative homotopy set is denoted by $\pi_1(X, Y, y_0)$.

DEFINITION 5.2. Let X be a topological space and $Y \subset X$ a subspace. For a map

$$(5.1) \quad \mu : \Omega(X, Y) \longrightarrow \mathbb{Z}$$

the properties *Concatenation* and *Homotopy* are defined as follows:

- (1) *Concatenation*: If $f, g \in \Omega(X, Y)$ are paths with $f(1) = g(0)$ then

$$(5.2) \quad \mu(f * g) = \mu(f) + \mu(g).$$

- (2) *Homotopy*: μ descends to a map $\mu : \tilde{\pi}_1(X, Y) \rightarrow \mathbb{Z}$.

LEMMA 5.3. Let X, Y be as before and let $\mu : \Omega(X, Y) \rightarrow \mathbb{Z}$ be a map which satisfies Concatenation and Homotopy.

Then for each $x_0 \in Y$ the restriction of μ to $\Omega(X, x_0)$ is a homomorphism $\pi_1(X, x_0) \rightarrow \mathbb{Z}$. Furthermore, for any path $f : [0, 1] \rightarrow Y$ we have $\mu(f) = 0$.

PROOF. The first claim is obvious. To prove the second claim we first note that a constant path $f = f(0)$ satisfies $\mu(f) = \mu(f * f) = \mu(f) + \mu(f)$, thus $\mu(f) = 0$.

A general path $f \in \Omega(Y)$ is homotopic in $\Omega(X, Y)$ to the constant path $f(0)$ via the homotopy $f_s(t) := f(st)$ and we reach the conclusion. \square

5.2. Warm-up: Uniqueness in the bounded case. Let H be a separable complex Hilbert space, i.e. the Hilbert dimension is finite or countably infinite. If $A \subset \mathcal{C}$ is a set of operators in H we denote by GA the set of invertible elements in A (with bounded inverse), by $\mathcal{F}A$ the Fredholm operators in A , and by \mathcal{F}_*A the Fredholm operators in A which are neither essentially positive nor essentially negative.

The following uniqueness-theorem for the spectral flow in the classical situation of bounded operators follows easily from the isomorphism (1.3). This is of course folklore.

THEOREM 5.4. Let H be an infinite-dimensional separable complex Hilbert space and let $\mu : \Omega(\mathcal{BF}_*^{\text{sa}}, G\mathcal{BF}_*^{\text{sa}}) \rightarrow \mathbb{Z}$ be a map which satisfies Concatenation, Homotopy and

Normalization: There is a $T_0 \in G\mathcal{BF}_*^{\text{sa}}$ and a rank one orthogonal projection $P \in \mathcal{B}^{\text{sa}}$ commuting with T_0 such that the path

$$f_P(t) := tP + (I - P)T_0, \quad -1/2 \leq t \leq 1/2$$

satisfies

$$\mu(f_P) = 1.$$

Then μ equals the spectral flow.

PROOF. We first note that the spectral flow satisfies *Concatenation*, *Homotopy*, and *Normalization*.

$G\mathcal{BF}_*^{\text{sa}}$ is connected. Therefore, we may choose a path $g_P \in \Omega(G\mathcal{BF}_*^{\text{sa}})$ from $f_P(1)$ to $f_P(0)$. In view of Lemma 5.3, and *Normalization* we thus have $\mu(g_P * f_P) = \text{SF}(g_P * f_P) = 1$. By (1.3) the closed path $g_P * f_P$ must be a generator of $\pi_1(\mathcal{BF}_*^{\text{sa}}, f_P(0))$ and consequently $\mu = \text{SF}$ on $\pi_1(\mathcal{BF}_*^{\text{sa}}, f_P(0))$ again by Lemma 5.3.

If $f \in \Omega(\mathcal{BF}_*^{\text{sa}}, G\mathcal{BF}_*^{\text{sa}})$ is arbitrary we choose paths $g_1, g_2 : [0, 1] \rightarrow G\mathcal{BF}_*^{\text{sa}}$ with $g_1(0) = f_P(0), g_1(1) = f(0), g_2(0) = f(1), g_2(1) = f_P(0)$. Then the path $g_1 * f * g_2$ is closed and Lemma 5.3 yields

$$(5.3) \quad \mu(f) = \mu(g_1 * f * g_2) = \text{SF}(g_1 * f * g_2) = \text{SF}(f)$$

and we are done. \square

Amazingly the finite-dimensional analogue of the previous Theorem is slightly more complicated due to the fact that in this case $G\mathcal{B}^{\text{sa}}$ is not connected. Of course, if H is finite-dimensional then $\mathcal{BF}^{\text{sa}} = \mathcal{B}^{\text{sa}}$.

PROPOSITION 5.5. *If $\dim H < \infty$ then the path components of $G\mathcal{B}^{\text{sa}}$ are labelled by $\text{rank}(1_{[0, \infty)}(T)) \in \{0, \dots, \dim H\}$.*

PROOF. $T \mapsto \text{rank}(1_{[0, \infty)}(T))$ is continuous on $G\mathcal{B}^{\text{sa}} \subset GL(\dim H)$ and maps onto $\{0, \dots, \dim H\}$.

Obviously, for $T \in G\mathcal{B}^{\text{sa}}$ there is a path in $G\mathcal{B}^{\text{sa}}$ connecting T with $2P - I$ for $P = 1_{[0, \infty)}(T)$ which shows injectivity. \square

LEMMA 5.6. *Let $\dim H < \infty$ and let $f, g \in \Omega(\mathcal{B}^{\text{sa}}, G\mathcal{B}^{\text{sa}})$ be paths with the same initial points $f(0) = g(0)$.*

Then f, g define the same class in $\tilde{\pi}_1(\mathcal{B}^{\text{sa}}, G\mathcal{B}^{\text{sa}})$ if and only if

$$(5.4) \quad \text{rank}(1_{[0, \infty)}(f(1))) = \text{rank}(1_{[0, \infty)}(g(1))).$$

PROOF. If f, g define the same class in $\tilde{\pi}_1(\mathcal{B}^{\text{sa}}, G\mathcal{B}^{\text{sa}})$ then $f(1)$ and $g(1)$ lie certainly in the same path component of $G\mathcal{B}^{\text{sa}}$ and hence (5.4) holds by Proposition 5.5.

The exact homotopy sequence of the pair $(\mathcal{B}^{\text{sa}}, G\mathcal{B}^{\text{sa}})$ gives a bijection

$$(5.5) \quad \pi_1(\mathcal{B}^{\text{sa}}, G\mathcal{B}^{\text{sa}}, f(0)) \rightarrow \pi_0(G\mathcal{B}^{\text{sa}}), \quad [h] \mapsto [h(1)],$$

hence from Proposition 5.5 we infer that f, g even define the same class in the relative homotopy set $\pi_1(X, Y, f(0))$, in particular they define the same class in $\tilde{\pi}_1(X, Y)$. \square

THEOREM 5.7. *Let H be a finite-dimensional Hilbert space and let*

$$\mu : \Omega(\mathcal{B}^{\text{sa}}, G\mathcal{B}^{\text{sa}}) \longrightarrow \mathbb{Z}$$

be a map which satisfies Concatenation, Homotopy and Normalization in the following sense:

There is a rank one orthogonal projection $P \in \mathcal{B}^{\text{sa}}$ such that for all $A \in \mathcal{B}^{\text{sa}}$

$$(5.6) \quad \mu((tP + (I - P)A(I - P))_{-1/2 \leq t \leq 1/2}) = 1.$$

Then

$$\mu(f) = \text{rank}(1_{[0, \infty)}(f(1))) - \text{rank}(1_{[0, \infty)}(f(0))) = \text{SF}(f)$$

for all $f \in \Omega(\mathcal{B}^{\text{sa}}, G\mathcal{B}^{\text{sa}})$.

PROOF. First note that *Normalization* holds for any rank one orthogonal projection: namely all rank one orthogonal projections are unitarily equivalent and the unitary group is connected, hence *Homotopy* implies that *Normalization* holds for any rank one orthogonal projection P .

Now consider a path $f \in \Omega(\mathcal{B}^{\text{sa}}, G\mathcal{B}^{\text{sa}})$. In view of Proposition 5.5 and *Homotopy* we may assume that

$$(5.7) \quad f(0) = 2P - I \text{ and } f(1) = 2Q - I,$$

where P, Q are orthogonal projections. Put

$$(5.8) \quad \gamma_P(t) = 2P - I + 2t(I - P), \quad 0 \leq t \leq 1.$$

Then $\gamma_P^- * f * \gamma_Q$ starts and ends at I .

By Lemma 5.6 $\gamma_P^- * f * \gamma_Q$ is homotopic in $\Omega(\mathcal{B}^{\text{sa}}, G\mathcal{B}^{\text{sa}})$ to the constant curve I , hence from *Concatenation*, *Homotopy* and Lemma 5.3 we infer that

$$(5.9) \quad \mu(f) = \mu(\gamma_P) - \mu(\gamma_Q).$$

If we can show that $\mu(\gamma_P) = n - \text{rank } P$ then we find

$$(5.10) \quad \mu(f) = \text{rank } Q - \text{rank } P = \text{rank}(1_{[0, \infty)}(f(1))) - \text{rank}(1_{[0, \infty)}(f(0))) = \text{SF}(f).$$

In the last equation we have used Corollary 3.7.

It remains to show $\mu(\gamma_P) = n - \text{rank } P$. Fix an orthonormal basis such that P has the matrix representation

$$(5.11) \quad P = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

Then γ_P is homotopic to $\gamma_1 * \gamma_2 * \dots * \gamma_{n-k}$ where

$$(5.12) \quad \gamma_j(t) = \begin{pmatrix} I_{k+j-1} & & \mathbf{0} \\ & 2t-1 & \\ \mathbf{0} & & -I_{n-k-j} \end{pmatrix}.$$

Normalization and the remark at the beginning of this proof show $\mu(\gamma_j) = 1$. Hence we find with *Concatenation* and *Homotopy*

$$(5.13) \quad \mu(\gamma_P) = \sum_{j=1}^{n-k} \mu(\gamma_j) = n - k = n - \text{rank } P. \quad \square$$

5.3. Uniqueness of the spectral flow for graph continuous paths. Let H be an infinite-dimensional separable complex Hilbert space. During this subsection we consider the *graph topology* on $\mathcal{C}\mathcal{F}^{\text{sa}}$.

We treat the bounded and the unbounded case simultaneously. Thereby we reprove Theorem 5.4 without using (1.3). We now let X be

$$\mathcal{B}\mathcal{F}_*^{\text{sa}} := \{T \in \mathcal{B} \mid T = T^*, T \text{ Fredholm}, \text{spec}_{\text{ess}} T \cap \mathbb{R}_{\pm} \neq \emptyset\} \text{ or } \mathcal{C}\mathcal{F}^{\text{sa}}.$$

X is connected since $\mathcal{B}\mathcal{F}_*^{\text{sa}}$ is trivially connected and $(\mathcal{C}\mathcal{F}^{\text{sa}}, d_G)$ is connected by [BBLP01, Thm. 1.10].

Next put $Y := GX$, i.e. $Y = G\mathcal{B}\mathcal{F}_*^{\text{sa}} = \mathcal{B}\mathcal{F}_*^{\text{sa}} \cap G\mathcal{B}^{\text{sa}}$ or $Y = G\mathcal{C}\mathcal{F}^{\text{sa}}$. Again, Y is connected. In the bounded case this is trivial. In the unbounded case it is less obvious:

PROPOSITION 5.8. *If $\dim H = \infty$ then $(G\mathcal{C}\mathcal{F}^{\text{sa}}, d_G)$ is path connected.*

This is proved in the spirit of [BBLP01, Thm. 1.10] and in fact we also reprove the connectedness of $\mathcal{C}\mathcal{F}^{\text{sa}}$ in a slightly different way. During the proof we use the notation of loc. cit. freely.

PROOF. We look at the Cayley picture and consider $U = \kappa(T)$. Recall that the Cayley transform κ is a homeomorphism from $G\mathcal{C}\mathcal{F}^{\text{sa}}$ onto

$$(5.14) \quad \kappa(G\mathcal{C}\mathcal{F}^{\text{sa}}) = \{U \in \mathcal{U} \mid U + I \text{ invertible and } U - I \text{ injective}\}.$$

As in loc. cit. $H = H_+ \oplus H_-$ is the direct sum of the spectral subspaces of U corresponding to $\{\lambda \in S^1 \mid \text{Im } \lambda \geq 0\}$ and $\{\lambda \in S^1 \mid \text{Im } \lambda < 0\}$ and by squeezing the spectrum down to $+i$ and $-i$ one can deform U within $\kappa(G\mathcal{C}\mathcal{F}^{\text{sa}})$ to

$$(5.15) \quad U_1 = +iI_+ \oplus -iI_-$$

(cf. Figure 2). Now since $\dim H = \infty$ we have $\dim H_+ = \infty$ or $\dim H_- = \infty$.

Case I: $\dim H_- = \infty$. As described in loc. cit. we may un-contract $-iI_-$ in such a way that no eigenvalues remain, i.e.

$$(5.16) \quad U_1 \sim iI_+ \oplus V_-,$$

where $\text{spec } V_-$ consists of a little arc centred on $-i$ and V_- has no eigenvalues. We then rotate this arc up through $+1$ until it is centered on $+i$. Then we contract the spectrum to be $+i$. This homotopy will stay within $\kappa(G\mathcal{C}\mathcal{F}^{\text{sa}})$ and deform U_1 to iI_H .

Case II: $\dim H_+ = \infty$. As in Case I we now un-contract $+iI_+$ and deform U_1 into $U_2 = -iI_H$. Now the operator U_2 has $\dim H_-(U_2) = \infty$. Applying Case I we deform U_2 to $+iI_H$. \square

Next we choose a base point $T_0 \in G\mathcal{B}\mathcal{F}_*^{\text{sa}} \subset Y$ with $\text{spec } T_0 = \{\pm 1\}$, in particular T_0 is bounded. With these preparations the uniqueness of the spectral flow on $\mathcal{C}\mathcal{F}^{\text{sa}}$ reads as follows:

THEOREM 5.9. *Let*

$$\mu : \Omega(X, Y) \longrightarrow \mathbb{Z}$$

be a map which satisfies Concatenation, Homotopy and Normalization in the following sense:

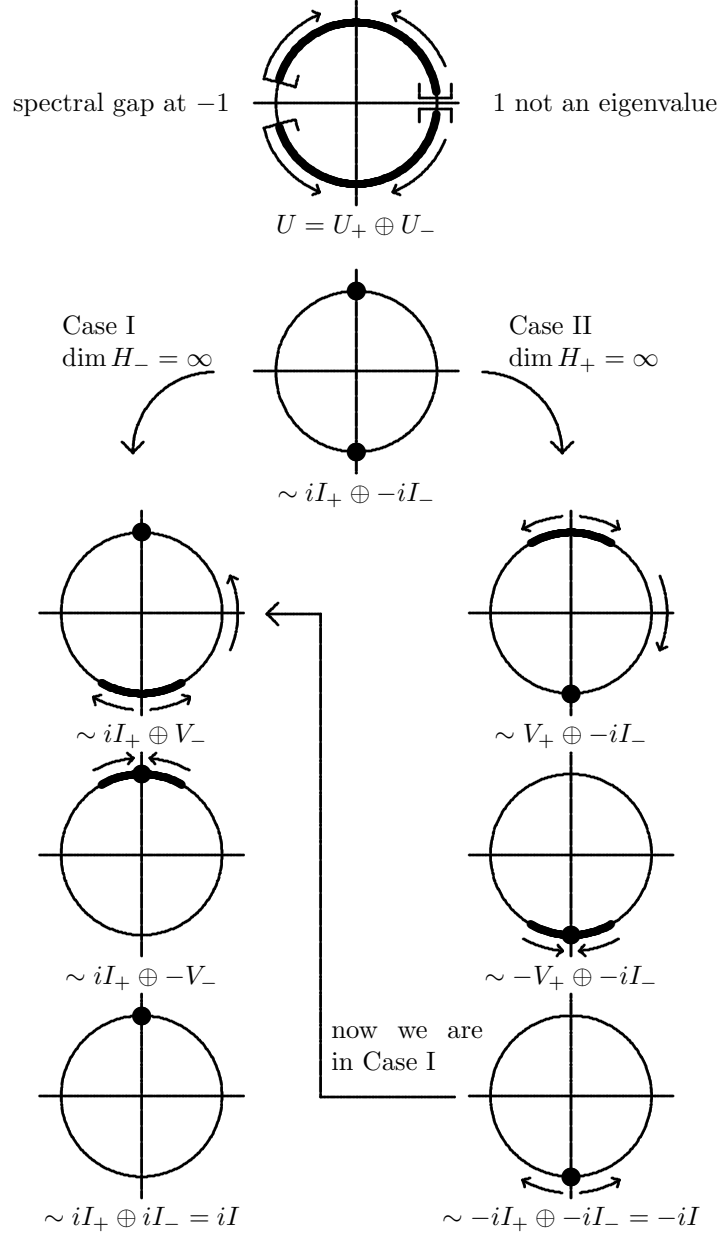


FIGURE 2. Connecting a fixed $U \in \kappa(G\mathcal{CF}^{\text{sa}})$ to iI . Case II (infinite rank U_+) is first deformed to $-iI$ and then Case I (infinite rank U_-) applies. Through the deformation -1 is never a spectral point.

There is a rank one orthogonal projection $P \in \mathcal{B}^{\text{sa}}$ such that the operator $(I - P)T_0(I - P) \in \mathcal{B}^{\text{sa}}(\ker P)$ is invertible and such that

$$(5.17) \quad \mu((tP + (I - P)T_0(I - P))_{-1/2 \leq t \leq 1/2}) = 1.$$

Then μ equals the spectral flow.

PROOF. We will deform a general path $f \in \Omega(X, Y)$ in several steps into some normal form and then compare μ and SF on this normal form. The latter problem is basically reduced to the finite-dimensional case which was treated as a warm-up in the previous subsection.

In the sequel \sim denotes homotopy in $\Omega(X, Y)$. Consider $f \in \Omega(X, Y)$.

ASSERTION 1. *There exist paths $f_1, \dots, f_n \in \Omega(X, Y)$ having the following properties*

- (1) $f \sim f_1 * \dots * f_n$.
- (2) *There exist $\varepsilon_j > 0$ such that for all $t \in [0, 1]$ we have $\pm \varepsilon_j \notin \text{spec } f_j(t)$ and $\text{spec}_{\text{ess}}(f_j(t)) \cap [-\varepsilon_j, \varepsilon_j] = \emptyset$.*

This is a basic fact about paths of Fredholm operators and has nothing to do with our assumptions on μ . Namely, for each t the operator $f(t)$ is Fredholm and hence $0 \notin \text{spec}_{\text{ess}} f(t)$. Hence by compactness (cf. [BBLP01, Prop. 2.10 and Def. 2.12]) there is a subdivision $0 = t_0 < t_1 < \dots < t_n = 1$ of the interval $[0, 1]$ and positive real numbers $\varepsilon_j, j = 1, \dots, n$, such that $\pm \varepsilon_j \notin \text{spec } f(t)$ and $[-\varepsilon_j, \varepsilon_j] \cap \text{spec}_{\text{ess}}(f(t)) = \emptyset$ for $t_{j-1} \leq t \leq t_j, j = 1, \dots, n$.

Hence the candidates for the f_j are $f|_{[t_{j-1}, t_j]}$ (reparametrized over $[0, 1]$). The problem is that $f(t_j)$ need not be invertible and hence need not be in Y . However, 0 is at most an isolated point of the spectrum of $f(t_j)$ since $f(t_j)$ is Fredholm. Hence there is a $\delta > 0$ such that $f(t_j) + s\delta$ is invertible for $0 < s \leq \delta, j = 1, \dots, n-1$ and $0 \leq s \leq \delta, j = 0, n$. By compactness we may choose δ so small that additionally $\pm \varepsilon_j \notin \text{spec}(f(t) + s\delta)$ and $[-\varepsilon_j, \varepsilon_j] \cap \text{spec}_{\text{ess}}(f(t) + s\delta) = \emptyset$ for all $t_{j-1} \leq t \leq t_j, 0 \leq s \leq \delta$. Thus

$$(5.18) \quad H(s, t) := f(t) + sI, \quad 0 \leq s \leq \delta$$

is a homotopy in $\Omega(X, Y)$ and we reach the conclusion with $f_j := f|_{[t_{j-1}, t_j]} + \delta I$ (reparametrized over $[0, 1]$).

ASSERTION 2. *Suppose that there is an $\varepsilon > 0$ such that $\pm \varepsilon \notin \text{spec}(f(t))$ and $[-\varepsilon, \varepsilon] \cap \text{spec}_{\text{ess}}(f(t)) = \emptyset$ for all $t \in [0, 1]$. Then f is homotopic in $\Omega(X, Y)$ to a path g having the following properties: there is a finite rank orthogonal projection \tilde{Q} and an operator $S \in G((I - \tilde{Q})X(I - \tilde{Q}))$ such that with respect to the orthogonal decomposition $H = \text{im } \tilde{Q} \oplus \ker \tilde{Q}$ we have*

$$(5.19) \quad g(t) = \begin{pmatrix} g_0(t) & 0 \\ 0 & S \end{pmatrix}.$$

Since $[-\varepsilon, \varepsilon] \cap \text{spec}_{\text{ess}}(f(t)) = \emptyset$ the spectral projection

$$(5.20) \quad E(t) := 1_{(-\varepsilon, \varepsilon)}(f(t))$$

is of finite rank and $E(t)$ depends continuously on t by [BBLP01, Prop. 2.10].

By [Bla86, Prop. 4.3.3] there exists a continuous family of unitaries $U : [0, 1] \rightarrow \mathcal{U}, U(0) = I$ such that $E(t) = U(t)E(0)U(t)^*$.

Now consider the homotopy

$$(5.21) \quad H(s, t) := U(st)^* f(t) U(st), \quad 0 \leq s, t \leq 1.$$

H is certainly a homotopy in $\Omega(X, Y)$. Furthermore,

$$(5.22) \quad \begin{aligned} H(1, t)E(0) &= U(t)^* f(t) U(t) E(0) = U(t)^* f(t) E(t) U(t) \\ &= U(t)^* E(t) f(t) U(t) = E(0) H(1, t), \end{aligned}$$

since $E(t)$ is a spectral projection of $f(t)$. Thus the orthogonal projection $E(0)$ commutes with the self-adjoint operator $H(1, t)$ and hence with respect to the decomposition $H = \text{im } E(0) \oplus \ker E(0)$ the operator $H(1, t)$ takes the form

$$(5.23) \quad H(1, t) = \begin{pmatrix} g_0(t) & 0 \\ 0 & g_1(t) \end{pmatrix}.$$

By construction $g_1(t)$ is invertible for all t and thus the map

$$(5.24) \quad \begin{pmatrix} g_0(t) & 0 \\ 0 & g_1(st) \end{pmatrix}, \quad 0 \leq s, t \leq 1$$

homotops $H(1, \cdot)$ to

$$(5.25) \quad \begin{pmatrix} g_0(t) & 0 \\ 0 & g_1(0) \end{pmatrix}, \quad 0 \leq s, t \leq 1,$$

and Assertion 2 is proved with $S = g_1(0)$ and $\tilde{Q} = E(0)$.

ASSERTION 3. *Consider g as in Assertion 2. Then g is homotopic in $\Omega(X, Y)$ to a path h having the following properties: there is a finite rank orthogonal projection $Q \geq P$ such that $(I - Q)T_0(I - Q) \in \mathcal{B}^{\text{sa}}(\ker Q)$ is invertible and such that with respect to the orthogonal decomposition $H = \text{im } Q \oplus \ker Q$ we have*

$$(5.26) \quad h(t) = \begin{pmatrix} h_0(t) & 0 \\ 0 & (I - Q)T_0(I - Q) \end{pmatrix}.$$

There is a unitary operator $U \in \mathcal{U}$ such that $Q := U^* \tilde{Q} U \geq P$ and $(I - Q)T_0(I - Q) \in \mathcal{B}^{\text{sa}}(\ker Q)$ is invertible. Since the unitary group \mathcal{U} is connected we may choose a path $U : [0, 1] \rightarrow \mathcal{U}$ with $U(1) = U, U(0) = I$. Then $H(s, t) := U(s)^* g(t) U(s)$ homotops g in $\Omega(X, Y)$ to $U^* g U$. W.r.t. the orthogonal decomposition $H = \text{im } Q \oplus \ker Q$ the latter takes the form

$$(5.27) \quad \tilde{h}(t) = \begin{pmatrix} h_0(t) & 0 \\ 0 & \tilde{S} \end{pmatrix}.$$

Finally, since $G((I - Q)X(I - Q))$ is path connected¹ there is a path $h_1 : [0, 1] \rightarrow G((I - Q)X(I - Q))$ with $h_1(0) = \tilde{S}$ and $h_1(1) = (I - Q)T_0(I - Q)$ and

$$(5.28) \quad \begin{pmatrix} h_0(t) & 0 \\ 0 & h_1(s) \end{pmatrix}, \quad 0 \leq s, t \leq 1$$

homotops \tilde{h} to the claimed path h .

¹Note that Q is of finite rank and hence $(I - Q)X(I - Q)$ is either $G\mathcal{BF}_*^{\text{sa}}(\ker Q)$ or $G\mathcal{C}^{\text{sa}}(\ker Q)$. In either case $G((I - Q)X(I - Q))$ is path connected since $\ker Q$ is a separable infinite-dimensional Hilbert space (Proposition 5.8).

FINISH OF PROOF. In view of *Homotopy* and *Concatenation* and in view of Assertions 1–3 it remains to show that for h in (5.26) we have $\mu(h) = \text{SF}(h)$.

Consider the map

$$(5.29) \quad \begin{aligned} \sigma : \Omega(\mathcal{B}^{\text{sa}}(\text{im } Q), G\mathcal{B}^{\text{sa}}(\text{im } Q)) &\longrightarrow \mathbb{Z}, \\ \sigma(f) &:= \mu\left(\begin{pmatrix} f & 0 \\ 0 & (I - Q)T_0(I - Q) \end{pmatrix}\right). \end{aligned}$$

σ inherits *Concatenation* and *Homotopy* immediately from μ . σ is also normalized since $P \leq Q$.

Thus we may apply Theorem 5.7 and conclude

$$(5.30) \quad \mu(h) = \sigma(h_0) = \text{rank}(1_{[0,\infty)}(h_0)) - \text{rank}(1_{[0,\infty)}(h_0)) = \text{SF}(h_0) = \text{SF}(h). \quad \square$$

5.4. Uniqueness of the spectral flow for Riesz continuous paths. Because of the next result all results about the spectral flow of paths of bounded operators carry over verbatim to *Riesz* continuous paths of unbounded operators. The drawback is, as mentioned in the introduction, that the Riesz metric is so strong that it is hard to prove continuity of maps into the space.

THEOREM 5.10. *The natural inclusion of the pair*

$$j : (\mathcal{B}\mathcal{F}^{\text{sa}}, G\mathcal{B}^{\text{sa}}) \hookrightarrow ((\mathcal{C}\mathcal{F}^{\text{sa}}, G\mathcal{C}^{\text{sa}}), d_R)$$

is a homotopy equivalence.

PROOF. The image of the Riesz map $F(T) = T(I + T^2)^{-1/2}$ was determined in [BBLP01, Prop. 1.5], i.e.

$$(5.31) \quad F(\mathcal{C}^{\text{sa}}) = \{S \in \mathcal{B}^{\text{sa}} \mid \|S\| \leq 1 \text{ and } S \pm I \text{ both injective}\} =: X.$$

F is a homeomorphism of \mathcal{C}^{sa} onto $X \subset \mathcal{B}^{\text{sa}}$ by definition of the Riesz metric. From the functional calculus we know that F maps the (essential) spectrum of T onto the (essential) spectrum of $F(T)$. Furthermore, F maps \mathcal{B}^{sa} onto the set $Y := \{S \in X \mid \|S\| < 1\} \subset X$.

Denoting by GX the invertible elements in X and by $\mathcal{F}X$ the Fredholm elements in X (and similarly for Y) we find that

$$(5.32) \quad F : ((\mathcal{C}\mathcal{F}^{\text{sa}}, G\mathcal{C}^{\text{sa}}), d_R) \longrightarrow ((\mathcal{F}X, GX), \|\cdot\|)$$

is a homeomorphism.

$F|_{\mathcal{B}^{\text{sa}}}$ is a homeomorphism onto Y , too. Namely, by [BBLP01, Prop. 1.5] the inverse of F is given by $F^{-1}(S) = (1 - S^2)^{-1/2}S$ and this is certainly norm continuous on Y . Hence

$$(5.33) \quad F : ((\mathcal{B}\mathcal{F}^{\text{sa}}, G\mathcal{B}^{\text{sa}}), \|\cdot\|) \longrightarrow ((\mathcal{F}Y, GY), \|\cdot\|)$$

is a homeomorphism, too.

In sum, it suffices to prove that the inclusion

$$(5.34) \quad \beta := F \circ j \circ F^{-1} : (\mathcal{F}Y, GY) \hookrightarrow (\mathcal{F}X, GX)$$

is a homotopy equivalence. Recall that we are now dealing with sets of *bounded* self-adjoint operators which are equipped with the usual norm topology. Therefore, the map

$$(5.35) \quad H : X \times [0, 1/2] \longrightarrow X, (S, t) \mapsto S(I + S^2)^{-t}$$

is trivially continuous. Moreover, it has the mapping properties

$$(5.36) \quad \begin{aligned} H(GX \times [0, 1/2]) &\subset GX, \\ H(Y \times [0, 1/2]) &\subset Y, \\ H(X \times (0, 1/2]) &\subset Y, \\ H(GY \times [0, 1/2]) &\subset GY. \end{aligned}$$

Put $g : (\mathcal{F}X, GX) \rightarrow (\mathcal{F}Y, GY)$, $g := H(\cdot, 1/2)$. Then g is a homotopy inverse of β since H is a homotopy between $\text{id}_{(\mathcal{F}X, GX)}$ and $\beta \circ g$ and the restriction of H to $Y \times [0, 1/2]$ is a homotopy between $\text{id}_{(\mathcal{F}Y, GY)}$ and $g \circ \beta$. \square

REMARK 5.11.

(1) Theorem 5.10 was observed in [Nic00, Sec. 3] without proof.

(2) We leave it to the reader to calculate the homotopy inverse $F^{-1} \circ g \circ F$ of j and the corresponding homotopy. It is a tedious formula. Intuitively one would try the map F itself to be a homotopy inverse of j and the homotopy to be (same formula as H) $(T, s) \mapsto T(I + T^2)^{-s}$, $0 \leq s \leq 1/2$. However, for unbounded T the operator $T(I + T^2)^{-s}$ is bounded for $s = 1/2$ and unbounded for $s < 1/2$ and proving continuity at $s = 1/2$ seems to be tedious, though we did not try very hard.

As a consequence of Theorem 5.10 we note:

COROLLARY 5.12. *Let H be an infinite-dimensional separable complex Hilbert space. Then $(\mathcal{C}\mathcal{F}_*^{\text{sa}}, d_R)$ is a classifying space for the K^1 -functor. Its homotopy groups are given by (1.2) and the uniqueness for the spectral flow holds as in Theorem 5.4.*

Recall that in subsection 5.3 we give a proof of Theorem 5.4 which is independent of [AS69].

5.5. Uniqueness of the spectral flow for the d_W -metric. For completeness we state the uniqueness for the spectral flow in $(\mathcal{F}\mathcal{B}^{\text{sa}}, d_W)$.

THEOREM 5.13. *Let H be an infinite-dimensional separable Hilbert space and let D be a fixed self-adjoint operator in H , $W := \mathcal{D}(D)$. Let*

$$\mu : \Omega(\mathcal{F}\mathcal{B}^{\text{sa}}(W, H), G\mathcal{B}^{\text{sa}}) \longrightarrow \mathbb{Z}$$

be a map which satisfies Concatenation, Homotopy and Normalization in the following sense:

There is a rank one orthogonal projection P with $\text{im } P \subset W$ such that for all $A \in \mathcal{B}^{\text{sa}}(W, H)$ with $(I - P)A(I - P)$ invertible we have

$$\mu((tP + (I - P)A(I - P))_{-1/2 \leq t \leq 1/2}) = 1.$$

The *Normalization* condition is slightly more complicated here. Superficially, this is because we have formulated the theorem for paths in $\mathcal{F}\mathcal{B}^{\text{sa}}(W, H)$ instead of $\mathcal{F}_*\mathcal{B}^{\text{sa}}(W, H)$. But this is not really the point. The problem is that we do not even know whether $\mathcal{F}_*\mathcal{B}^{\text{sa}}(W, H)$ is path connected or not. If it is then *Normalization* can be formulated as in Theorem 5.9.

Theorem 5.13 is basically due to Robbin and Salamon [RS95], who assumed additionally that D has compact resolvent. The formulation in loc. cit. is slightly different since they impose a *Direct Sum* axiom.

The proof of Theorem 5.13 just follows along the lines of the proof of Theorem 5.4: Assertions 1 and 2 just carry over word by word. At first glance the family of unitaries chosen in the proof of Assertion 2 might be problematic. However, since $E(t)$ is finite-rank with image in W one sees that $U(t)$ can be chosen as a finite-rank perturbation of I and such that U maps W into itself. The proof of Assertion 3 uses that $(G\mathcal{C}^{\text{sa}}, d_G)$ is path connected. Here this is taken care of by the stronger *Normalization* condition which allows to skip Assertion 3 and go directly to the “Finish of Proof” on page 27. We leave the details to the reader.

Appendix A. Some estimates

In this appendix we collect a couple of operator estimates which are basically well-known but for which references are hard to find.

PROPOSITION A.1. *Let H be a separable Hilbert space and let T be an (unbounded) self-adjoint operator in H with bounded inverse. Furthermore, let B be a symmetric operator in H with $\mathcal{D}(B) \supset \mathcal{D}(T)$.*

- (1) $T^{-1}BT$ is densely defined and $(T^{-1}BT)^* = TBT^{-1}$.
- (2) If $T^{-1}BT$ or TBT^{-1} is densely defined and bounded then B, TBT^{-1} and $T^{-1}BT$ are densely defined and bounded and we have $\|TBT^{-1}\| = \|T^{-1}BT\|$ and $\|B\| \leq \|T^{-1}BT\|$.
- (3) If $T^{-1}B$ is bounded then so is $T^{-1/2}BT^{-1/2}$ and $\|T^{-1/2}BT^{-1/2}\| \leq \|T^{-1}B\|$.

Note that, by definition,

$$(A.1) \quad \begin{aligned} \mathcal{D}(TBT^{-1}) &= \{x \in H \mid BT^{-1}x \in \mathcal{D}(T)\} \\ \mathcal{D}(T^{-1}BT) &= \{x \in \mathcal{D}(T) \mid Tx \in \mathcal{D}(B)\}. \end{aligned}$$

In (1) it is *not* claimed that $\mathcal{D}(TBT^{-1})$ is dense. Hence if $T^{-1}BT$ is bounded then it *follows* that TBT^{-1} is defined on H and bounded. Note that by (1) the operator TBT^{-1} is always closed.

PROOF. This is basically a consequence of complex interpolation theory (Taylor [Tay96, Sec. 4.2]) but we prefer to give a direct elementary proof here.

We first note that (3) follows from (2): namely the operator $X := T^{-1/2}BT^{-1/2}$ is symmetric on $\mathcal{D}(T^{1/2})$ and $T^{-1/2}XT^{1/2}$ is densely defined and bounded. Now apply (2) with $B = X$ and $T^{1/2}$ instead of T .

To prove (1) we note that certainly $\mathcal{D}(T^{-1}BT) = \{x \in \mathcal{D}(T) \mid Tx \in \mathcal{D}(B)\} \supset \mathcal{D}(T^2)$. Since T is self-adjoint $\mathcal{D}(T^2)$ is a core for T (i.e. $\mathcal{D}(T^2)$ is dense in $\mathcal{D}(T)$ with respect to the graph norm), in particular it is dense in H . Thus $T^{-1}BT$ is densely defined.

Now consider $x \in \mathcal{D}(TBT^{-1})$ and $y \in \mathcal{D}(T^{-1}BT)$. Then, by (A.1), $y \in \mathcal{D}(T)$, $Ty \in \mathcal{D}(B)$ and $T^{-1}x \in \mathcal{D}(B)$, $BT^{-1} \in \mathcal{D}(T)$ and consequently,

$$(A.2) \quad \langle TBT^{-1}x, y \rangle = \langle BT^{-1}x, Ty \rangle = \langle T^{-1}x, BTy \rangle = \langle x, T^{-1}BTy \rangle.$$

This shows $TBT^{-1} \subset (T^{-1}BT)^*$.

To show the converse inclusion, let us consider $x \in \mathcal{D}((T^{-1}BT)^*)$ and $y \in \mathcal{D}(T^2) \subset \mathcal{D}(T^{-1}BT)$. Carefully checking domains we find

$$(A.3) \quad \langle (T^{-1}BT)^*x, y \rangle = \langle x, T^{-1}BTy \rangle = \langle T^{-1}x, BTy \rangle = \langle BT^{-1}x, Ty \rangle.$$

As noted above, $\mathcal{D}(T^2)$ is dense in $\mathcal{D}(T)$ with respect to the graph norm. Hence the equality

$$(A.4) \quad \langle (T^{-1}BT)^*x, y \rangle = \langle BT^{-1}x, Ty \rangle$$

holds for all $y \in \mathcal{D}(T)$. But this means that $BT^{-1}x \in \mathcal{D}(T)$ and $TBT^{-1}x = (T^{-1}BT)^*x$ proving $TBT^{-1} \supset (T^{-1}BT)^*$.

To prove (2) assume that the densely defined operator $T^{-1}BT$ is bounded. Then its adjoint $TBT^{-1} = (T^{-1}BT)^*$ is densely defined and bounded, too.

If TBT^{-1} is densely defined and bounded on H then we infer from $TBT^{-1} = (T^{-1}BT)^*$ that $T^{-1}BT$ is closable and $\overline{T^{-1}BT} = (TBT^{-1})^*$. Hence $T^{-1}BT$ is (densely defined and) bounded.

It is now clear that if TBT^{-1} and $T^{-1}BT$ are bounded that then $\|TBT^{-1}\| = \|T^{-1}BT\|$.

Now suppose that they are both bounded and pick $x, y \in \mathcal{D}(T^2)$ and consider the analytic function

$$(A.5) \quad f(z) := \langle T^{2z-1}BT^{1-2z}x, y \rangle, \quad 0 < \operatorname{Re} z < 1.$$

Since $x, y \in \mathcal{D}(T^2)$ it is straightforward to check that f is bounded and continuous on the vertical strip $\{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1\}$. Moreover, we have for $z = it, t \in \mathbb{R}$,

$$(A.6) \quad |f(z)| = |\langle T^{-1}BT T^{-2it}x, T^{-2it}y \rangle| \leq \|T^{-1}BT\| \|x\| \|y\|,$$

and similarly for $z = 1 + it, t \in \mathbb{R}$,

$$(A.7) \quad |f(z)| = |\langle T^{-2it}x, T^{-1}BT^{1-2it}y \rangle| \leq \|T^{-1}BT\| \|x\| \|y\|.$$

Hence by Hadamard's three line theorem (Rudin [Rud87, Thm. 12.8]) we find $|f(z)| \leq \|T^{-1}BT\| \|x\| \|y\|$ for $0 \leq \operatorname{Re} z \leq 1$. In particular we have for $z = 1/2$

$$(A.8) \quad |\langle Bx, y \rangle| \leq \|T^{-1}BT\| \|x\| \|y\|, \quad x, y \in \mathcal{D}(T^2).$$

Since $\mathcal{D}(T^2)$ is dense in H we reach the conclusion. \square

PROPOSITION A.2. *Let $T \in \mathcal{C}^{\text{sa}}, S \in \mathcal{B}^{\text{sa}}$ with $(T + i)S(T + i)^{-1}$ densely defined and bounded. Then, for $0 \leq \alpha \leq 2, -1 \leq \beta \leq 1, \alpha + \beta < 2$ we have the norm estimate*

$$\begin{aligned} & \left\| |T + S + i|^\alpha (|T + S + i|^{-1} - |T + i|^{-1}) |T + i|^\beta \right\| \\ & \leq C(\alpha, \beta) \left(\|(T + i)S(T + i)^{-1}\| + \|S^2(T + i)^{-1}\| \right). \end{aligned}$$

REMARK A.3. For $(T + S + i)^{-1} - (T + i)^{-1}$ in place of $|T + S + i|^{-1} - |T + i|^{-1}$ the estimate follows easily from the resolvent equation

$$(A.9) \quad (T + S + i)^{-1} - (T + i)^{-1} = -(T + S + i)^{-1}S(T + i)^{-1}$$

and complex interpolation theory.

To deal with the operator absolute value recall that for any non-negative invertible operator A in H one has

$$(A.10) \quad A^{-1/2} = \frac{2}{\pi} \int_0^\infty (A + x^2)^{-1} dx.$$

PROOF. Note that $(T + i)S(T + i)^{-1}$ bounded means that S maps $\mathcal{D}(T)$ continuously into $\mathcal{D}(T)$. Hence $\mathcal{D}((T + S)^2) \supset \mathcal{D}(T^2)$ and thus $(T + S)^2 - T^2 = (T + S)S + ST$ on $\mathcal{D}(T^2)$. Thus we have for $x \geq 0$ the resolvent identity

$$\begin{aligned} & (I + (T + S)^2 + x^2)^{-1} - (I + T^2 + x^2)^{-1} \\ (A.11) \quad &= -(I + (T + S)^2 + x^2)^{-1} ((T + S)S + ST)(I + T^2 + x^2)^{-1} \\ &=: \mathcal{J}(T, S, x), \end{aligned}$$

and in view of (A.10) we find

$$\begin{aligned} & (|T + S + i|^{-1} - |T + i|^{-1}) \\ (A.12) \quad &= -\frac{2}{\pi} \int_0^\infty \mathcal{J}(T, S, x) dx. \end{aligned}$$

Next we estimate the integrand of (A.12)

$$\begin{aligned} & \left\| |T + S + i|^\alpha \mathcal{J}(T, S, x) |T + i|^\beta \right\| \\ & \leq \left\| |T + S + i|^\alpha (I + (T + S)^2 + x^2)^{-1} \right\| \cdot \dots \\ & \quad \cdot \left\| (T + S)S |T + i|^{-1} \right\| \left\| |T + i| (I + T^2 + x^2)^{-1} |T + i|^\beta \right\| \\ (A.13) \quad & + \left\| |T + S + i|^\alpha (I + (T + S)^2 + x^2)^{-1} \right\| \cdot \dots \\ & \quad \cdot \|S\| \left\| T(I + T^2 + x^2)^{-1} |T + i|^\beta \right\| \\ & \leq C(\alpha, \beta) (1 + x^2)^{(\alpha + \beta - 3)/2} (\|S\| + \|(T + S)S |T + i|^{-1}\|) \\ & \leq C(\alpha, \beta) (1 + x^2)^{(\alpha + \beta - 3)/2} (\| |T + i|S |T + i|^{-1} \| + \|S^2 |T + i|^{-1}\|). \end{aligned}$$

In the last inequality we have used Proposition A.1.

If $\alpha + \beta < 2$ we may integrate (A.13) and reach the conclusion. \square

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